9.3 The Integral Test

7. $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^{3/2}}$. Let $f(x) = \frac{x}{(x^2+1)^{3/2}}$. Then f is nonnegative, continuous, and decreasing on $[1, \infty)$.

$I = \int_1^{\infty} \frac{x\,dx}{(x^2+1)^{3/2}} = \lim_{b\to\infty} \int_1^b x\left(x^2+1\right)^{-3/2} dx = \lim_{b\to\infty} \left[-\left(x^2+1\right)^{-1/2}\right]_1^b = \lim_{b\to\infty} \left(-\frac{1}{\sqrt{b^2+1}} + \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$.

Since I is convergent, so is the series.

27. $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)} = \sum_{n=1}^{\infty} \left(\frac{-\frac{1}{2}}{2n+1} + \frac{\frac{1}{2}}{2n-1}\right)$ is a telescoping series:

$S_n = \sum_{k=1}^{n} \left(\frac{-\frac{1}{2}}{2k+1} + \frac{\frac{1}{2}}{2k-1}\right) = \frac{1}{2}\left[\left(-\frac{1}{3}+1\right) + \left(-\frac{1}{5}+\frac{1}{3}\right) + \left(-\frac{1}{7}+\frac{1}{5}\right) + \cdots + \left(-\frac{1}{2n+1} + \frac{1}{2n-1}\right)\right]$

$= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) \to \frac{1}{2}$ as $n \to \infty$

and so $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$ converges.

28. $\sum_{n=1}^{\infty} \frac{n}{2^n}$. Let $f(x) = \frac{x}{2^x}$. Then f is nonnegative, continuous, and decreasing on $[2, \infty)$. To find $I = \int_1^{\infty} x2^{-x}\,dx$, let $u = x$ and $dv = 2^{-x}\,dx \Rightarrow du = dx$ and $v = -\frac{2^{-x}}{\ln 2}$. Then

$I = -\frac{x}{2^x \ln 2} + \frac{1}{\ln 2}\int 2^{-x}\,dx = -\frac{x}{2^x \ln 2} - \frac{1}{2^x (\ln 2)^2} + C = -\left(x + \frac{1}{\ln 2}\right)\frac{1}{2^x \ln 2} + C$, so

$\int_1^{\infty} x2^{-x}\,dx = \lim_{b\to\infty} I = \lim_{b\to\infty} \left[-\left(b + \frac{1}{\ln 2}\right)\left(\frac{1}{2^b \ln 2}\right) + \left(1 + \frac{1}{\ln 2}\right)\left(\frac{1}{2\ln 2}\right)\right] = \left(1 + \frac{1}{\ln 2}\right)\left(\frac{1}{2\ln 2}\right)$. Thus, the series converges.

31. $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+5} = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2+4}$. Let $f(x) = \frac{1}{(x+1)^2+4}$.

Then f is nonnegative, continuous, and decreasing on $[1, \infty)$.

$I = \int_1^{\infty} \frac{dx}{(x+1)^2+4} = \lim_{b\to\infty} \int_1^b \frac{dx}{(x+1)^2+4} = \lim_{b\to\infty} \left[\frac{1}{2}\tan^{-1}\frac{b+1}{2} - \frac{1}{2}\tan^{-1}\frac{1+1}{2}\right] = \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}$,

showing that the series converges.

9.4 The Comparison Tests

5. $\frac{1}{\sqrt{n^2-1}} > \frac{1}{\sqrt{n^2}} = \frac{1}{n}$ for $n \geq 2$. Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, so does $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$.

15. If n is large, then $a_n = \frac{n}{\sqrt{n^5-1}}$ behaves like $\frac{n}{\sqrt{n^5}} = \frac{1}{n^{3/2}} = b_n$.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n}{\sqrt{n^5-1}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^{5/2}}{\sqrt{n^5-1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1-(1/n^5)}} = 1 > 0$, so $\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ converges $\Rightarrow \sum_{n=2}^{\infty} \frac{n}{\sqrt{n^5-1}}$ converges.

25. $\sum_{n=1}^{\infty} \frac{n+1}{(n+2)(2n^2+1)}$. If n is large, then $a_n = \frac{n+1}{(n+2)(2n^2+1)}$ behaves like $\frac{n}{n(2n^2)} = \frac{1}{2n^2}$, so we take $b_n = \frac{1}{n^2}$.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{n+1}{(n+2)(2n^2+1)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3+n^2}{(n+2)(2n^2+1)} = \frac{1}{2} > 0$, so $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{n+1}{(n+2)(2n^2+1)}$ converges.

38. $a_n = \frac{1}{1+2+3+\cdots+n} = \frac{1}{\frac{n(n+1)}{2}} = \frac{2}{n(n+1)} < \frac{2}{n^2}$. Since $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges, so does $\sum_{n=1}^{\infty} \frac{1}{1+2+3+\cdots+n}$.