聯合微積分 第三次作業解答

2.1 The Derivative

Use the definition of the derivative to find the derivative of the function.What is its domain?

8. $f(x) = 2\sqrt{x}$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{2\sqrt{x+h}-2\sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{2(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{2[(\sqrt{x+h})^2-(\sqrt{x})^2]}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{2[(x+h)-x]}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{2h}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{2}{\sqrt{x+h}+\sqrt{x}} = \frac{2}{2\sqrt{x}} = \frac{1}{\sqrt{x}} \\
&= x^{-\frac{1}{2}}
\end{aligned}$$

Domain: $(-\infty, 0) \cup (0, \infty)$ or $\{x \in \Re : x \neq 0\}$

12. $f(x) = -\frac{2}{\sqrt{x}}$

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-\frac{2}{\sqrt{x+h}} - (-\frac{2}{\sqrt{x}})}{h} \\
&= \lim_{h \to 0} \frac{\frac{2(\sqrt{x+h} - \sqrt{x})}{\sqrt{x+h}\sqrt{x}}}{h} = \lim_{h \to 0} \frac{2(\sqrt{x+h} - \sqrt{x})}{h\sqrt{x+h}\sqrt{x}} \\
&= \lim_{h \to 0} \frac{2(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h\sqrt{x}\sqrt{x+h}(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{2[(x+h) - x]}{h\sqrt{x}\sqrt{x+h}(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{2h}{h\sqrt{x}\sqrt{x+h}(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{2}{\sqrt{x}\sqrt{x+h}(\sqrt{x+h} + \sqrt{x})} \\
&= \frac{2}{\sqrt{x}\sqrt{x}(2\sqrt{x})} = \frac{1}{x(\sqrt{x})} \\
&= x^{-\frac{3}{2}}
\end{aligned}
$$

Domain: $(0, \infty)$ or $\{x \in \Re : x > 0\}$

Find an equation of the tangent line to the graph of the function at the indicated point.

18. $f(x) = 3x^3 - x$, $(-1, -2)$

檢驗 $f(-1) = 3(-1)^3 - (-1) = -2$ ，因此點 $(-1, -2)$ 在圖形上

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[3(x+h)^3 - (x+h)] - (3x^3 - x)}{h} \\
&= \lim_{h \to 0} \frac{[3(x+h)^3 - 3x^3] - [(x+h) - x]}{h} \\
&= \lim_{h \to 0} \frac{3[(x+h) - x][(x+h)^2 + x(x+h) + x^2] - h}{h} \\
&= \lim_{h \to 0} \frac{3h[(x+h)^2 + x(x+h) + x^2] - h}{h} \\
&= \lim_{h \to 0} \{3[(x+h)^2 + x(x+h) + x^2] - 1\} \\
&= 9x^2 - 1
\end{aligned}
$$

$f'(-1)$ 為函數 $f(x)$ 過點 $(-1, -2)$ 的切線斜率

$x=-1$ 代入 $f'(x)=9x^2-1$ 得到 $f'(-1)=9(-1)^2-1=8$

Ans: $y-(-2)=8[x-(-1)]$ or $y=8x+6$

Find the rate of change of each function with respect to x at the given value of x.

26. $y=x^2-\frac{1}{x}$; $x=-1$

設 $y=f(x)=x^2-\frac{1}{x}$

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{[(x+h)^2-\frac{1}{x+h}]-(x^2-\frac{1}{x})}{h}\\ &=\lim_{h\to0}\frac{[(x+h)^2-x^2]-(\frac{1}{x+h}-\frac{1}{x})}{h}=\lim_{h\to0}\frac{[(x+h)-x][(x+h)+x]+(\frac{(x+h)-x}{x(x+h)})}{h}\\ &=\lim_{h\to0}\frac{h(2x+h)+(\frac{h}{x(x+h)})}{h}=\lim_{h\to0}[(2x+h)+\frac{1}{x(x+h)}]\\ &=2x+\frac{1}{x^2}\end{aligned}$$

$\frac{dy}{dx}|_{x=-1}=f'(-1)=2(-1)+\frac{1}{(-1)^2}=-1$

Ans: -1

Show that the function is continuous but not differentiable at the given value of x.

50. $f(x)=\begin{cases} x+1, & \text{if } x\le 0\\ x^2+1, & \text{if } x>0\end{cases}$; $x=0$

$$\begin{aligned}&\lim_{x\to0^+}f(x)=\lim_{x\to0^+}(x^2+1)=1\\ =&\lim_{x\to0^-}f(x)=\lim_{x\to0^-}(x+1)=1\end{aligned}$$

$\lim_{x\to0}f(x)$ 存在，並且 $\lim_{x\to0}f(x)=1$ 又 $f(0)=0+1=1$

我們得到 $\lim_{x\to0}f(x)=f(0)$ 因此 $f(x)$ 在 $x=0$ 連續.

再來檢驗 $f(x)$ 在 $x=0$ 可不可以微分

$$\lim_{h\to0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to0^+}\frac{(h^2+1)-1}{h}=\lim_{h\to0^+}\frac{h^2}{h}=\lim_{h\to0^+}h=0$$

$$\lim_{h\to0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to0^-}\frac{(h+1)-1}{h}=\lim_{h\to0^-}\frac{h}{h}=\lim_{h\to0^-}1=1$$

得到
$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}\neq\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}$$
因此 $f(x)$ 在 $x=0$ 不可微分

2.2 Basic Rules of Differentiation

Find the derivative of the function.

10. $f(x)=0.3x^{-1.2}$

$$f'(x)=\frac{d}{dx}(0.3x^{-1.2})=-1.2(0.3x^{-1.2-1})=-0.36x^{-2.2}$$

13. $f(r)=\pi r^2+2\pi r$

$$\begin{aligned}f'(r)&=\frac{d}{dr}(\pi r^2+2\pi r)=\frac{d}{dr}(\pi r^2)+\frac{d}{dr}(2\pi r)\\&=2\pi r^{2-1}+2\pi r^{1-1}\\&=2\pi r+2\pi\end{aligned}$$

22. $f(x)=5x^{\frac{4}{3}}-\frac{2}{3}x^{\frac{3}{2}}+x^2-3x+1$

$$\begin{aligned}f'(x)&=\frac{d}{dx}(5x^{\frac{4}{3}}-\frac{2}{3}x^{\frac{3}{2}}+x^2-3x+1)\\&=\frac{d}{dx}(5x^{\frac{4}{3}})-\frac{d}{dx}(\frac{2}{3}x^{\frac{3}{2}})+\frac{d}{dx}(x^2)-\frac{d}{dx}(3x)+\frac{d}{dx}(1)\\&=\frac{4}{3}(5x^{\frac{4}{3}-1})-\frac{3}{2}(\frac{2}{3}x^{\frac{3}{2}-1})+2x^{2-1}-3x^{1-1}\\&=\frac{20}{3}x^{\frac{1}{3}}-x^{\frac{1}{2}}+2x-3\end{aligned}$$

71. Determine the constants A,B, and C such that the parabola $y=Ax^2+Bx+C$ passes through the point$(-1,0)$ and is tangent to the line $y=x$ at the point where $x=1$.

將 $(-1,0)$ 代入 $y=Ax^2+Bx+C$ 得到

$$0=A(-1)^2+B(-1)+C$$

$$A-B+C=0$$

再來 $y'=2Ax+B$ 由於拋物線 $y=Ax^2+Bx+C$ 在 $x=1$ 的切線 $y=x$ 斜率為 1 , 得到

$$2A+B=1$$

且點 $(1,1)$ 也在拋物線 $y = Ax^2 + Bx + C$ 上得到

$$A + B + C = 1$$

$$\begin{cases} A - B + C = 0 \\ A + B + C = 1 \\ 2A + B = 1 \end{cases}$$

$(2) - (1) \Rightarrow 2B = 1 \Rightarrow B = \frac{1}{2}$

$2A + \frac{1}{2} = 1 \Rightarrow 2A = \frac{1}{2} \Rightarrow A = \frac{1}{4}$

$\frac{1}{4} + \frac{1}{2} + C = 1 \Rightarrow C = \frac{1}{4}$

Ans: $A = \frac{1}{4}, B = \frac{1}{2}, C = \frac{1}{4}$

72. Let

$$f(x) = \begin{cases} x^2, & \text{if } x \le a \\ Ax + B, & \text{if } x > a \end{cases}$$

Find the values of A and B such that f is continuous and differentiable at a.

$f(a) = a^2$ 且 $\lim_{x \to a^-} f(x) = \lim_{x \to a^-} x^2 = a^2$

$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} Ax + B = Aa + B$

$Aa + B = a^2$

$$\begin{aligned} \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} &= \lim_{h \to 0^-} \frac{(a+h)^2 - a^2}{h} \\ &= \lim_{h \to 0^-} \frac{h(2a+h)}{h} \\ &= \lim_{h \to 0^-} (2a+h) \\ &= 2a \end{aligned}$$

$$\begin{aligned} \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} &= \lim_{h \to 0^+} \frac{[A(a+h) + B] - (Aa + B)}{h} \\ &= \lim_{h \to 0^+} \frac{Ah}{h} \\ &= \lim_{h \to 0^+} A \\ &= A \end{aligned}$$

$A = 2a, B = -a^2$

2.3 The Product and Quotient Rules

Find the derivative of each function in two ways.

16. $f(t) = \frac{2t^2 - 3t^{\frac{3}{2}}}{5t^{\frac{1}{2}}}$

< 方法一 >

$$f'(t) = \frac{d}{dt}(\frac{2}{5}t^{\frac{3}{2}} - \frac{3}{5}t) = \frac{3}{5}t^{\frac{1}{2}} - \frac{3}{5}$$

< 方法二 >

$$\begin{aligned} f'(t) &= \frac{(4t - \frac{9}{2}t^{\frac{1}{2}})(5t^{\frac{1}{2}}) - (2t^2 - 3t^{\frac{3}{2}})(\frac{5}{2}t^{-\frac{1}{2}})}{25t} \\ &= \frac{20t^{\frac{3}{2}} - \frac{45}{2}t - 5t^{\frac{3}{2}} + \frac{15}{2}t}{25t} \\ &= \frac{15t^{\frac{3}{2}} - 15t}{25t} \\ &= \frac{3}{5}t^{\frac{1}{2}} - \frac{3}{5} \end{aligned}$$

20. Find the derivative of each function.

$$f(x) = \frac{\sqrt{x} - 1}{\sqrt{x} + 1}$$

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\frac{\sqrt{x} - 1}{\sqrt{x} + 1}) \\ &= \frac{(\frac{1}{2}x^{-\frac{1}{2}})(\sqrt{x} + 1) - (\frac{1}{2}x^{-\frac{1}{2}})(\sqrt{x} - 1)}{(\sqrt{x} + 1)^2} \\ &= \frac{x^{-\frac{1}{2}}}{(\sqrt{x} + 1)^2} \end{aligned}$$

35. Find derivative of each function and evaluate $f'(x)$ at the given value of x.

$$f(x) = (\sqrt{x} + 2x)(x^{\frac{3}{2}} - x)\ ; x = 4$$

$$f'(x) = (\frac{1}{2}x^{-\frac{1}{2}} + 2)(x^{\frac{3}{2}} - x) + (\sqrt{x} + 2x)(\frac{3}{2}x^{\frac{1}{2}} - 1)$$

$$f'(4) = (\frac{1}{4} + 2)(8 - 4) + (2 + 8)(3 - 1) = 29$$

60. Find y''

$y = \frac{x}{x^2+1}$

$$y' = \frac{d}{dx}(\frac{x}{x^2+1}) = \frac{(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

$$\begin{aligned} y'' &= \frac{d}{dx}(\frac{1-x^2}{(x^2+1)^2}) = \frac{-2x(x^2+1)^2 - (1-x^2)2(2x)(x^2+1)}{(x^2+1)^4} \\ &= \frac{-2x}{(x^2+1)^2} + \frac{4x(x^2-1)}{(x^2+1)^3} \\ &= \frac{2x(x^2-3)}{(x^2+1)^3} \end{aligned}$$

62. Find

a. $f'''(0)$ if $f(x) = 8x^7 - 6x^5 + 4x^3 - x$

$f'(x) = 56x^6 - 30x^4 + 12x^2 - 1$

$f''(x) = 336x^5 - 120x^3 + 24x$

$f'''(x) = 1680x^4 - 360x^2 + 24$

Ans: $f'''(0) = 24$

b. $y'''|_{x=1}$ if $y = x^{-1}$

$y' = -x^{-2}$

$y'' = 2x^{-3}$

$y''' = -6x^{-4}$

Ans:$y'''|_{x=1} = -6$