3.3 Concept Questions

1. See page 267.

2. See page 268.

3. See page 271.

3.3 Increasing and Decreasing Functions and the First Derivative Test

1. **a.** f is increasing on $(-\infty, -2)$, constant on $(-2, 2)$, and decreasing on $(2, \infty)$.

b. f has a relative maximum value of 2 attained at all the values of x on the interval $[-2, 2]$; f has no relative minimum.

2. **a.** f is increasing on $(-\infty, \infty)$.

b. f has no relative extremum.

3. **a.** f is decreasing on $(-\infty, -1)$ and increasing on $(-1, \infty)$.

b. f has a relative minimum value of 0 attained at -1.

4. **a.** f is decreasing on $(-\infty, -1)$ and $(1, \infty)$ and increasing on $(-1, 1)$.

b. f has a relative minimum value of $-\frac{1}{2}$ attained at -1 and a relative maximum value of $\frac{1}{2}$ attained at 1.

5. **a.** f is increasing on $(-\infty, -1)$ and $(-1, \infty)$.

b. f has no relative extremum.

6. **a.** f is increasing on $(-\infty, -2)$ and $(-2, 0)$ and decreasing on $(0, 2)$ and $(2, \infty)$.

b. f has a relative maximum value of -2 attained at 0.

7. **a.** $f'(x) < 0$ on approximately $(-\infty, -2.5)$ and $(2.5, \infty)$. Therefore, f is decreasing on those intervals. $f'(x) > 0$ on approximately $(-2.5, 2.5)$, so f is increasing on this interval.

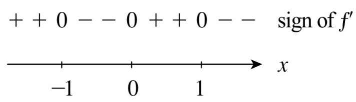
b. $f'(x) = 0$ at $x \approx -2.5$ and $x \approx 2.5$, and these are critical numbers of f. Since $f'(x) < 0$ if $x < -2.5$, and $f'(x) > 0$ if $x > -2.5$, we see that f has a relative minimum at -2.5. Next, $f'(x) > 0$ if $x < 2.5$ and $f'(x) < 0$ if $x > 2.5$, and so f has a relative maximum at 2.5.

8. **a.** $f'(x) < 0$ on $(-\infty, -2)$, and so f is decreasing on $(-\infty, -2)$. $f'(x) > 0$ on $(-2, 0)$ and $(0, \infty)$, and so f is increasing on these intervals.

b. $f'(x) = 0$ at $x = -2$, and this is a critical number of f. Since $f'(x) < 0$ for $x < -2$ and $f'(x) > 0$ for $x > -2$, we see that $x = -2$ gives a relative minimum.

16. $f(x) = -x^4 + 2x^2 + 1 \Rightarrow$

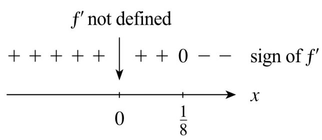
$f'(x) = -4x^3 + 4x = -4x\left(x^2 - 1\right) = -4x(x+1)(x-1)$ is continuous everywhere and has zeros at -1, 0, and 1, the critical numbers of f. The sign diagram of f' is shown.



a. f is increasing on $(-\infty, -1)$ and $(0, 1)$ and decreasing on $(-1, 0)$ and $(1, \infty)$.

b. f has relative maxima of $f(-1) = f(1) = 2$ and a relative minimum of $f(0) = 1$.

18. $f(x) = x^{1/3} - x^{2/3} \Rightarrow f'(x) = \frac{1}{3}x^{-2/3} - \frac{2}{3}x^{-1/3} = \frac{1}{3}x^{-2/3}\left(1 - 2x^{1/3}\right)$ is discontinuous at 0 and has a zero at $x = \frac{1}{8}$. The critical numbers of f are thus 0 and $\frac{1}{8}$. The sign diagram of f' is shown.



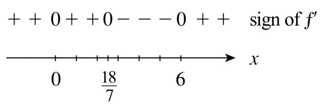
a. f is increasing on $\left(-\infty, \frac{1}{8}\right)$ and decreasing on $\left(\frac{1}{8}, \infty\right)$.

b. f has a relative maximum of $f\left(\frac{1}{8}\right) = \frac{1}{4}$.

20. $f(x) = x^3(x-6)^4 \Rightarrow$

$$f'(x) = 3x^2(x-6)^4 + x^3(4)(x-6)^3 = x^2(x-6)^3[3(x-6) + 4x]$$
$$= x^2(7x - 18)(x-6)^3$$

is continuous everywhere and has zeros at 0, $\frac{18}{7}$, and 6, the critical numbers of f. The sign diagram of f' is shown.



a. f is increasing on $\left(-\infty, \frac{18}{7}\right)$ and $(6, \infty)$ and decreasing on $\left(\frac{18}{7}, 6\right)$.

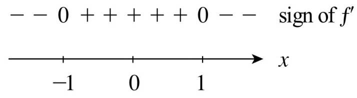
b. f has a relative maximum of $f\left(\frac{18}{7}\right) \approx 2350$ and a relative minimum of $f(6) = 0$.

22. $f(x) = \dfrac{x}{x-1} \Rightarrow f'(x) = \dfrac{(x-1)(1) - x(1)}{(x-1)^2} = -\dfrac{1}{(x-1)^2}$ is discontinuous at 1 and has no zeros, but 1 is not in the domain of f, so f has no critical numbers. Furthermore, $f'(x)$ is negative on its domain.

a. f is decreasing on $(-\infty, 1)$ and $(1, \infty)$.

b. f has no relative extremum.

24. $f(x) = \dfrac{x}{x^2+1} \Rightarrow$

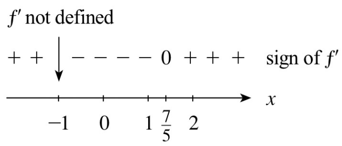


$f'(x) = \dfrac{(x^2+1)(1) - x(2x)}{\left(x^2+1\right)^2} = \dfrac{1-x^2}{\left(x^2+1\right)^2} = \dfrac{(1-x)(1+x)}{\left(x^2+1\right)^2}$ is continuous everywhere and has zeros at ± 1, the critical numbers of f. The sign diagram of f' is shown.

a. f is decreasing on $(-\infty, -1)$ and $(1, \infty)$ and increasing on $(-1, 1)$.

b. f has a relative minimum of $f(-1) = -\frac{1}{2}$ and a relative maximum of $f(1) = \frac{1}{2}$.

26. $f(x) = \dfrac{x^2-3x+2}{x^2+2x+1} = \dfrac{x^2-3x+2}{(x+1)^2} = \dfrac{(x-2)(x-1)}{(x+1)^2} \Rightarrow$



$$f'(x) = \frac{(x+1)^2(2x-3) - (x^2-3x+2)2(x+1)}{(x+1)^4}$$

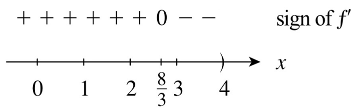
$$= \frac{(x+1)\left[(x+1)(2x-3) - 2(x^2-3x+2)\right]}{(x+1)^4} = \frac{5x-7}{(x+1)^3}$$

is discontinuous at $x = -1$ and has a zero at $\frac{7}{5}$. Since -1 is not in the domain of f, the only critical number is $\frac{7}{5}$. The sign diagram of f' is shown.

a. f is decreasing on $\left(-1, \frac{7}{5}\right)$ and increasing on $(-\infty, -1)$ and $\left(\frac{7}{5}, \infty\right)$.

b. f has a relative minimum of $f\left(\frac{7}{5}\right) = -\frac{1}{24}$.

28. $f(x) = x\sqrt{4-x} = x(4-x)^{1/2} \Rightarrow$



$$f'(x) = (4-x)^{1/2} + x\left(\tfrac{1}{2}\right)(4-x)^{-1/2}(-1)$$

$$= -\tfrac{1}{2}(4-x)^{-1/2}\left[-2(4-x) + x\right] = -\frac{(3x-8)}{2\sqrt{4-x}}$$

is discontinuous at $x = 4$ and has a zero at $\frac{8}{3}$, both critical numbers of f. The sign diagram of f' is shown.

a. f is increasing on $\left(-\infty, \frac{8}{3}\right)$ and decreasing on $\left(\frac{8}{3}, 4\right)$.

b. f has a relative maximum of $f\left(\frac{8}{3}\right) \approx 3.08$.

30. $f(x) = \dfrac{x}{\sqrt{x^2-1}} = \dfrac{x}{\left(x^2-1\right)^{1/2}}$; the domain of f is $(-\infty, -1) \cup (1, \infty)$.

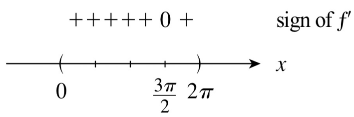
$$f'(x) = \frac{d}{dx}\left[x\left(x^2-1\right)^{-1/2}\right] = \left(x^2-1\right)^{-1/2} + x\left(-\tfrac{1}{2}\right)\left(x^2-1\right)^{-3/2}(2x) = \left(x^2-1\right)^{-1/2} - x^2\left(x^2-1\right)^{-3/2}$$

$= \left(x^2-1\right)^{-3/2}\left[\left(x^2-1\right) - x^2\right] = -\dfrac{1}{\left(x^2-1\right)^{3/2}}$ has no zero and is negative on its domain.

a. f is decreasing on $(-\infty - 1)$ and $(1, \infty)$.

b. f has no relative extremum.

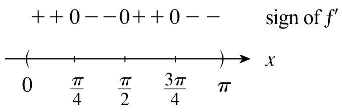
32. $f(x) = x - \cos x$, $0 < x < 2\pi \Rightarrow f'(x) = 1 + \sin x$ is continuous on $(0, 2\pi)$ and has zeros where $1 + \sin x = 0 \Leftrightarrow \sin x = -1 \Rightarrow x = \frac{3\pi}{2}$, a critical number of f. The sign diagram of f' is shown.



a. f is increasing on $(0, 2\pi)$.

b. f has no relative extremum.

34. $f(x) = \sin^2 2x,\ 0 < x < \pi \Rightarrow$

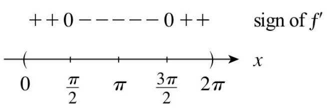


$f'(x) = 2(\sin 2x \cos 2x)(2) = 4 \sin 2x \cos 2x = 2 \sin 4x$ is continuous and has zeros where $\sin 4x = 0 \Rightarrow x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ in $(0, \pi)$. The sign diagram of f' is shown.

a. f is increasing on $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$ and decreasing on $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{4}, \pi\right)$.

b. f has relative maxima of $f\left(\frac{\pi}{4}\right) = f\left(\frac{3\pi}{4}\right) = 1$ and a relative minimum of $f\left(\frac{\pi}{2}\right) = 0$.

36. $f(x) = \dfrac{\sin x}{1 + \sin^2 x},\ 0 < x < 2\pi \Rightarrow$



$$f'(x) = \frac{\left(1 + \sin^2 x\right)\cos x - (\sin x)(2 \sin x)\cos x}{\left(1 + \sin^2 x\right)^2} = \frac{\left(1 - \sin^2 x\right)\cos x}{\left(1 + \sin^2 x\right)^2} = \frac{\cos^3 x}{\left(1 + \sin^2 x\right)^2}$$

is continuous everywhere and has zeros at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, critical numbers of f. The sign diagram of f' is shown.

a. f is increasing on $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$ and decreasing on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

b. f has a relative maximum of $f\left(\frac{\pi}{2}\right) = \frac{1}{2}$ and a relative minimum of $f\left(\frac{3\pi}{2}\right) = -\frac{1}{2}$.

54. $f(x) = \tan x - x \Rightarrow f'(x) = \sec^2 x - 1 > 0$ for x in $\left(0, \frac{\pi}{2}\right) \Rightarrow f$ is increasing on $\left(0, \frac{\pi}{2}\right)$. Since $f(0) = 0$, we have $f(x) > 0$ for x in $\left(0, \frac{\pi}{2}\right)$, and so $\tan x - x > 0 \Rightarrow \tan x > x$ for x in $\left(0, \frac{\pi}{2}\right)$.

56. $f(x) = -2x^2 + ax + b \Rightarrow f'(x) = -4x + a$. We require that $f'(2) = 0 \Leftrightarrow -4 \cdot 2 + a = 0 \Rightarrow a = 8$, and $f(2) = -2 \cdot 2^2 + 8 \cdot 2 + b = 4 \Rightarrow b = -4$. Thus, $f(x) = -2x^2 + 8x - 4$. Since the graph of f is a parabola that opens downward, $(2, 4)$ is an absolute maximum.

66. True. Let $h(x) = f(x) + g(x)$, and let x_1 and x_2 be any two numbers in I with $x_1 < x_2$. Then $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$; that is, $f(x_2) - f(x_1) > 0$ and $g(x_2) - g(x_1) > 0$. Now $h(x_2) - h(x_1) = [f(x_2) + g(x_2)] - [f(x_1) - g(x_1)] = [f(x_2) - f(x_1)] + [g(x_2) - g(x_1)] > 0 \Rightarrow h(x_2) > h(x_1)$ $\Rightarrow h$ is increasing on I.

67. True. Let $h(x) = f(x) - g(x)$, and let x_1 and x_2 be any two numbers in I with $x_1 < x_2$. Then $f(x_1) < f(x_2) \Rightarrow f(x_2) - f(x_1) > 0$. Also, $g(x_2) < g(x_1) \Rightarrow g(x_1) - g(x_2) > 0$. Now $h(x_2) - h(x_1) = [f(x_2) - g(x_2)] - [f(x_1) - g(x_1)] = [f(x_2) - f(x_1)] + [g(x_1) - g(x_2)] > 0 \Rightarrow h$ is increasing on I.

68. False. Take $f(x) = g(x) = x$. Then f and g are increasing on $(-\infty, \infty)$, but $(fg)(x) = x^2$ is not increasing on $(-\infty, \infty)$.

69. True. Let $h = f/g$ and let x_1 and x_2 be any two numbers in I with $x_1 < x_2$. Then $f(x_2) - f(x_1) > 0$ and $g(x_1) - g(x_2) > 0$. Now

$$h(x_2) - h(x_1) = \frac{f(x_2)}{g(x_2)} - \frac{f(x_1)}{g(x_1)} = \frac{f(x_2)g(x_1) - g(x_2)f(x_1)}{g(x_2)g(x_1)}$$
$$= \frac{f(x_2)g(x_1) - f(x_1)g(x_1) + f(x_1)g(x_1) - g(x_2)f(x_1)}{g(x_2)g(x_1)}$$
$$= \frac{[f(x_2) - f(x_1)]g(x_1) + [g(x_1) - g(x_2)]f(x_1)}{g(x_2)g(x_1)} > 0 \Rightarrow h = f/g \text{ is increasing on } I.$$

70. False. Consider $f(x) = \begin{cases} x & \text{if } x < 0 \\ \frac{1}{2}x & \text{if } x \geq 0 \end{cases}$ Then f is increasing on $(-\infty, \infty)$, but $f'(0)$ does not exist.

71. False. Consider $f(x) = 2x$ and $g(x) = x + 10$ on $[0, 2]$. Then $f'(x) = 2 > 1 = g'(x)$. But $f(1) = 2 < 11 = g(1)$.

3.4 Concept Questions

1. See page 277.
2. See page 279.
3. See pages 283 and 284.

3.4 Concavity and Inflection Points

1. f is concave upward on $(0, \infty)$ and concave downward on $(-\infty, 0)$; it has an inflection point at $(0, 0)$.
2. f is concave downward on $(-\infty, \infty)$. It has no inflection point.
3. f is concave upward on $(-\infty, -4)$ and $(4, \infty)$ and concave downward on $(-4, 4)$. It has no inflection point.
4. f is concave upward on $(-\infty, 0)$ and $(1, \infty)$ and concave downward on $(0, 1)$. It has inflection points at $(0, 0)$ and $(1, -1)$.
5. f is concave downward on $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$. It has no inflection point.
6. f is concave upward on $(0, \infty)$ and concave downward on $(-\infty, 0)$. It has an inflection point at $(0, 1)$.

38. $h(x) = 2x^3 + 3x^2 - 12x - 2 \Rightarrow h'(x) = 6x^2 + 6x - 12 = 6(x+2)(x-1) = 0 \Rightarrow x = -2$ or 1, the critical numbers of h. $h''(x) = 12x + 6 = 6(2x+1)$. We use the SDT: $h''(-2) = -18 < 0$, so h has a relative maximum of $h(-2) = 18$; and $h''(1) = 18 > 0$, so h has a relative minimum of $h(1) = -9$.

40. $f(x) = 2x^4 - 8x + 4 \Rightarrow f'(x) = 8x^3 - 8 = 8(x-1)\left(x^2 + x + 1\right) = 0 \Rightarrow x = 1$, the sole critical number of f. $f''(x) = 24x^2$, and by the SDT, $f''(1) = 24 > 0$ shows that f has a relative minimum of $f(1) = -2$.

42. $h(t) = t^2 + \dfrac{1}{t} \Rightarrow h'(t) = 2t - t^{-2} = \dfrac{2t^3 - 1}{t^2} = 0 \Rightarrow t = \frac{\sqrt[3]{4}}{2}$, the only critical number of h. $h''(t) = 2 + \dfrac{2}{t^3}$, and we use the SDT: $h''\left(\frac{\sqrt[3]{4}}{2}\right) > 0$, so h has a relative minimum of $h\left(\frac{\sqrt[3]{4}}{2}\right) = \frac{3\sqrt[3]{2}}{2}$.

44. $f(x) = x\sqrt{4 - x^2} \Rightarrow f'(x) = \frac{1}{2}x\left(4 - x^2\right)^{1/2}(-2x) + \left(4 - x^2\right)^{1/2} = \dfrac{2\left(2 - x^2\right)}{\left(4 - x^2\right)^{1/2}} = 0 \Rightarrow x = \pm\sqrt{2}$, the critical numbers of f. Note that f' is not defined at $x = \pm 2$, but these are the endpoints of the domain of f. $f''(x) = 2\left[\dfrac{\left(4 - x^2\right)^{1/2}(-2x) - \left(2 - x^2\right)\frac{1}{2}\left(4 - x^2\right)^{-1/2}(-2x)}{4 - x^2}\right] = \dfrac{2x\left(x^2 - 6\right)}{\left(4 - x^2\right)^{3/2}}$. We use the SDT: $f''\left(-\sqrt{2}\right) = 4 > 0$, so f has a relative minimum of $f\left(-\sqrt{2}\right) = -2$; and $f''\left(\sqrt{2}\right) = -2 < 0$, so f has a relative maximum of $f\left(\sqrt{2}\right) = 2$.

46. $f(x) = \sin^2 x$, $0 < x < \frac{3\pi}{2} \Rightarrow f'(x) = 2\sin x\cos x = \sin 2x = 0 \Rightarrow x = \frac{\pi}{2}$ or π in $\left(0, \frac{3\pi}{2}\right)$, so these are critical numbers. $f''(x) = 2\cos 2x$. Using the SDT, we find that $f''\left(\frac{\pi}{2}\right) = -2 < 0$, so f has a relative maximum of $f\left(\frac{\pi}{2}\right) = 1$; and $f''(\pi) = 2 > 0$, so f has a relative minimum of $f(\pi) = 0$.

48. $h(t) = \dfrac{1}{1 - \cos t}$, $0 < t < 2\pi \Rightarrow h'(t) = \dfrac{d}{dt}(1 - \cos t)^{-1} = -(1 - \cos t)^{-2}\sin t = -\dfrac{\sin t}{(1 - \cos t)^2} = 0 \Rightarrow t = \pi$, the only critical number in $(0, 2\pi)$. $h''(t) = \dfrac{-(1 - \cos t)^2\cos t + (\sin t)(2)(1 - \cos t)\sin t}{(1 - \cos t)^4}$ and $h''(\pi) = \frac{4}{2^4} = \frac{1}{4} > 0$, so h has a relative minimum of $h(\pi) = \frac{1}{2}$.

77. False. For example, $f(x) = x^{1/3}$ has an inflection point at 0, but $f'(0)$ is not defined.

78. False. Consider $f(x) = \begin{cases} x^3 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ Then f is not continuous at $x = 0$, and so $(0, 1)$ cannot be an inflection point of f.

79. True. If f is a polynomial of degree 3, then f'' must be a linear function which has exactly one zero. Also, the sign of f'' must change as we move across that zero.

80. True. Since f is concave upward on I, $f''(x) > 0$ for all x in I. If $h = -f$, then $h''(x) = -f''(x) < 0$ for all x in I, and this shows that h is concave downward, so $-f$ is concave downward.