9.3

4. $\sum_{n=1}^{\infty} ne^{-n}$. Let $f(x) = xe^{-x}$. Then f is nonnegative, continuous, and decreasing on $[1, \infty)$.
$\int_1^{\infty} xe^{-x}\,dx = \lim_{b\to\infty} \int_1^b xe^{-x}\,dx = \lim_{b\to\infty} \left[-(x+1)e^{-x}\right]_1^b = \lim_{b\to\infty}\left[-(b+1)e^{-b} + 2e^{-1}\right] = 2/e$ (using l'Hôpital's Rule). Since $\int_1^{\infty} xe^{-x}\,dx$ converges, so does the series.

15. $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$. Let $f(x) = \frac{1}{\sqrt{x+1}}$. Then f is nonnegative, continuous, and decreasing on $[0, \infty)$.

$I = \int_0^{\infty} \frac{dx}{\sqrt{x+1}} = \lim_{b\to\infty} \int_0^b (x+1)^{-1/2}\,dx = \lim_{b\to\infty}\left(2\sqrt{b+1} - 2\sqrt{2}\right) = \infty$. Since I diverges, so does the series.

23. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$. Let $f(x) = \frac{1}{x(\ln x)^2}$. Then f is nonnegative, continuous, and decreasing on $[2, \infty)$.

$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty} \int_2^b \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty}\left[-\frac{1}{\ln b} + \frac{1}{\ln 2}\right] = \frac{1}{\ln 2}$, so the series converges.

25. $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n^2}$. Let $f(x) = \frac{\sin(1/x)}{x^2}$. Then f is nonnegative, continuous, and decreasing on $[1, \infty)$.

$\int_1^{\infty} \frac{\sin(1/x)}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\sin(1/x)}{x^2}\,dx = \lim_{b\to\infty}\left[\cos(1/b) - \cos 1\right] = 1 - \cos 1$, so the series is convergent.

34. $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$. If $p < 0$, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n^p} = \infty$, and so the series is divergent. For $p > 0$, let

$f(x) = \frac{\ln x}{x^p}$. Then f is nonnegative, continuous, and decreasing on $[1, \infty)$. Using the table of integrals, we find

$\int_1^{\infty} \frac{\ln x}{x^p}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^p}\,dx = \lim_{b\to\infty}\left\{\frac{b^{1-p}}{(1-p)^2}\left[-1 + (1-p)\ln b\right] + \frac{1}{(1-p)^2}\right\}$ for $p \neq 1$. If $p > 1$, then the integral converges, and if $p = 1$, then it diverges (see Exercise 21). If $0 \leq p < 1$, then the integral also diverges. Thus, the integral is convergent only for $p > 1$, as is the series.

9. 4

6. $\dfrac{1}{\sqrt{n^3+1}} < \dfrac{1}{\sqrt{n^3}} = \dfrac{1}{n^{3/2}}$ for $n \geq 1$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, so does $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^3+1}}$.

17. If n is large, then $a_n = \dfrac{3n^2+1}{2n^5+n+2}$ behaves like $\dfrac{3n^2}{2n^5} = \dfrac{3}{2n^3}$, so take $b_n = \dfrac{1}{n^3}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{3n^2+1}{2n^5+n+2}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{3n^5+n^3}{2n^5+n+2} = \frac{3}{2} > 0, \text{ so } \sum_{n=1}^{\infty} \frac{1}{n^3} \text{ converges} \Rightarrow \sum_{n=1}^{\infty} \frac{3n^2+1}{2n^5+n+2} \text{ converges.}$$

27. $\dfrac{n-1}{n^3+2} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$ for $n \geq 1$. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^3+2}$ converges.

38. $a_n = \dfrac{1}{1+2+3+\cdots+n} = \dfrac{1}{\frac{n(n+1)}{2}} = \dfrac{2}{n(n+1)} < \dfrac{2}{n^2}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges, so does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+2+3+\cdots+n}$.

47. Since $\sum b_n$ converges, $\lim_{n\to\infty} b_n = 0$. Therefore there exists a positive integer N such that $n \geq N \Rightarrow b_n < 1$. So for $n \geq N$, we have $a_n b_n < a_n$. Since $\sum a_n$ is convergent, so is $\sum a_n b_n$ by the Comparison Test.