3.1

38. $g'(t) = \frac{d}{dt}\left(4t^{1/3} + 3t^{4/3}\right) = \frac{4}{3}t^{-2/3} + 4t^{1/3} = \frac{4}{3}t^{-2/3}(1 + 3t) = \frac{4(3t+1)}{3t^{2/3}}$. $g'(t)$ is discontinuous at $t = 0$ and has a zero at $t = -\frac{1}{3}$. Since both are in the domain of g, the critical numbers of g are $-\frac{1}{3}$ and 0.

43. $f'(x) = \frac{d}{dx}\left(x^2 - x - 2\right) = 2x - 1$ is continuous everywhere and has a zero at $x = \frac{1}{2}$, giving $\frac{1}{2}$ as the only critical number of f in the interval. Testing the values of $f(x)$ at the endpoints and the critical number, we find that $f(0) = -2$, $f\left(\frac{1}{2}\right) = -\frac{9}{4}$, and $f(2) = 0$, so f has an absolute minimum value of $-\frac{9}{4}$ attained at $x = \frac{1}{2}$ and an absolute maximum value of 0 attained at $x = 2$.

52. $f(x) = 2x + \frac{1}{x}$ is discontinuous at $x = 0$, which lies on the interval $(-1, 3)$. Since $\lim_{x \to 0^-}\left(2x + \frac{1}{x}\right) = -\infty$ and $\lim_{x \to 0^+}\left(2x + \frac{1}{x}\right) = \infty$, we see that f has no absolute minimum or absolute maximum value.

53. $f'(x) = \frac{d}{dx}\left(x - 2x^{1/2}\right) = 1 - x^{-1/2} = 1 - \frac{1}{x^{1/2}} = \frac{x^{1/2} - 1}{x^{1/2}}$ has 1 as a critical number in the interval $(0, 9)$. $f(0) = 0$, $f(1) = -1$, and $f(9) = 3$, so f has an absolute minimum value of -1 attained at $x = 1$ and an absolute maximum value of 3 attained at $x = 9$.

58. $g'(x) = \frac{d}{dx}(\cos x - \sin x) = -\sin x - \cos x$ is continuous everywhere and has zeros where $-\sin x - \cos x = 0 \Leftrightarrow \tan x = -1 \Leftrightarrow x = \frac{3\pi}{4}$ or $x = \frac{7\pi}{4}$ on the interval $(0, 2\pi)$. $g(0) = 1$, $g\left(\frac{3\pi}{4}\right) = -\sqrt{2}$, $g\left(\frac{7\pi}{4}\right) = \sqrt{2}$, and $g(2\pi) = 1$, so g has an absolute minimum value of $-\sqrt{2}$ attained at $x = \frac{3\pi}{4}$, and an absolute maximum value of $\sqrt{2}$ attained at $x = \frac{7\pi}{4}$.

3. 2

22. $f(x) = 1 - x^{2/3} \Rightarrow f'(x) = -\frac{2}{3}x^{-1/3} = -\dfrac{2}{3x^{1/3}}$. Suppose there is a number c satisfying $-1 < c < 8$ such that

$f'(c) = \dfrac{f(b) - f(a)}{b - a} = \dfrac{f(8) - f(-1)}{8 - (-1)}$. Then $-\dfrac{2}{3c^{1/3}} = \dfrac{\left(1 - 8^{2/3}\right) - (1 - 1)}{9} = -\dfrac{1}{3} \Leftrightarrow c^{1/3} = 2 \Leftrightarrow c = 8$.

This contradiction shows that no such c exists. The result does not contradict the Mean Value Theorem because f is not differentiable on the interval $(-1, 8)$.

23. f does not satisfy the hypotheses of the Mean Value Theorem on $[0, 2]$ because it is not differentiable on $(0, 2)$. In fact, f fails to be differentiable at $x = 1$. To see this, we calculate

$\lim\limits_{h\to 0^+} \dfrac{f(1) - f(1-h)}{h} = \lim\limits_{h\to 0^+} \dfrac{1 - (1-h)^2}{h} = \lim\limits_{h\to 0^+} \dfrac{2h - h^2}{h} = \lim\limits_{h\to 0^+} (2 - h) = 2$ and

$\lim\limits_{h\to 0^-} \dfrac{f(1) - f(1-h)}{h} = \lim\limits_{h\to 0^-} \dfrac{2 - (1+h) - 1}{h} = \lim\limits_{h\to 0^-} \dfrac{-h}{h} = \lim\limits_{h\to 0^-} (-1) = -1$. Thus, the left-hand limit of the difference quotient is not equal to the right-hand limit, so $f'(1)$ does not exist.

24. $g(x) = x^4 - 2x^2 + x$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Furthermore, $g(0) = 0 = g(1)$. Therefore, by Rolle's Theorem, there exists at least one number c in $(0, 1)$ such that $g'(c) = 4c^3 - 4c + 1 = 0$. But $g'(x) = f(x)$, and so $f(c) = 0$, showing that f has at least one zero in $(0, 1)$.

33. Let $f(x) = \cos^2 x - \frac{1}{2}\cos 2x$. Observe that f is continuous and differentiable on $(-\infty, \infty)$.

$f'(x) = 2\cos x(-\sin x) - \frac{1}{2}(-\sin 2x)(2) = -2\sin x\cos x + \sin 2x = -\sin 2x + \sin 2x = 0$. Using Theorem 3, we see that $f(x) = C$ where C is a constant; that is, $f(x) = \cos^2 x - \frac{1}{2}\cos 2x = C$. To find C, we calculate $f\left(\frac{\pi}{2}\right) = 0 - \frac{1}{2}(-1) = C \Leftrightarrow C = \frac{1}{2}$, so $\cos^2 x - \frac{1}{2}\cos 2x = \frac{1}{2} \Leftrightarrow \cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

40. **a.** By the Mean Value Theorem, there exists a number c in (a, b) such that $\dfrac{f(b) - f(a)}{b - a} = f'(c)$. Since $a < c < b$, we have $a - a < c - a < b - a \Rightarrow 0 < c - a < b - a \Rightarrow 0 < \dfrac{c - a}{b - a} < 1$. Put $\theta = \dfrac{c - a}{b - a}$ and $h = b - a$. Then we have $c = a + (b - a)\theta = a + \theta h$, and so $\dfrac{f(b) - f(a)}{b - a} = f'(a + \theta h)$, $0 < \theta < 1$.

b. $\dfrac{f(a+h) - f(a)}{h} = \dfrac{(a+h)^2 - a^2}{h} = \dfrac{a^2 + 2ah + h^2 - a^2}{h} = 2a + h$ and $f'(a + \theta h) = 2x|_{x=a+\theta h} = 2a + 2\theta h$, so $2a + h = 2a + 2\theta h \Rightarrow \theta = \frac{1}{2}$.