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3.
$$\frac{dw}{dt} = \frac{\partial w}{\partial r} \frac{dr}{dt} + \frac{\partial w}{\partial s} \frac{ds}{dt} = (\cos s + s \cos r) (-2e^{-2t}) + (-r \sin s + \sin r) (3t^2 - 2)$$

$$= -2(\cos s + s \cos r) e^{-2t} + (\sin r - r \sin s) (3t^2 - 2)$$
4.
$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} = \frac{1}{x+y^2} \sec^2 t + \frac{2y}{x+y^2} \sec t \tan t = \frac{\sec t (\sec t + 2y \tan t)}{x+y^2}$$
9.
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} = (3x^2)(2u) + (3y^2)(2v) = 6(x^2u + y^2v) \text{ and}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} = (3x^2)(2v) + (3y^2)(2u) = 6(x^2v + y^2u)$$
11.
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} = (e^x \cos y) \left(\frac{2u}{u^2+v^2} \right) + (-e^x \sin y) \left(\frac{1}{2} \frac{\sqrt{uv}}{u} \right) = e^x \left(\frac{2u \cos y}{u^2+v^2} - \frac{\sqrt{uv} \sin y}{2u} \right) \text{ and}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} = (e^x \cos y) \left(\frac{2v}{u^2+v^2} \right) + (-e^x \sin y) \left(\frac{1}{2} \frac{\sqrt{uv}}{v} \right) = e^x \left(\frac{2v \cos y}{u^2+v^2} - \frac{\sqrt{uv} \sin y}{2v} \right).$$
25.
$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r} = \frac{2xy}{z^2} e^{st} + \frac{x^2}{z^2} s t e^{rt} + \left(-\frac{2x^2y}{z^3} \right) s t e^{rst} \text{ and}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t} = \frac{2xy}{z^2} r s e^{st} + \frac{x^2}{z^2} r s e^{rt} + \left(-\frac{2x^2y}{z^3} \right) r s e^{rst}.$$

If $r = 1$, $s = 2$, and $t = 0$, then $x = 1$, $y = 2$, and $z = 1$, so $\frac{\partial w}{\partial r} = 4 + 0 + (-4)(0) = 4$ and $\frac{\partial w}{\partial t} = 4(2) + 1(2) - 4(2) = 2$.
29. Here $F(x, y) = x^3 - 2xy + y^3 - 4 = 0$, so $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 2y}{-2x + 3y^2} = \frac{3x^2 - 2y}{2x - 3y^2}$.
33. Here $F(x, y, z) = x^2 + xy - x^2z + yz^2 = 0$, so $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x + y - 2xz}{-x^2 + 2yz} = \frac{2x + y - 2xz}{x^2 - 2yz}$ and
- $$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x + z^2}{-x^2 + 2yz} = \frac{x + z^2}{x^2 - 2yz}.$$
35. Here $F(x, y, z) = x e^y + y e^{xz} + x^2 e^{x/y} - 10 = 0$, so
- $$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{e^y + y z e^{xz} + \left(2x + \frac{x^2}{y}\right) e^{x/y}}{x y e^{xz}} = -\frac{y e^y + y^2 z e^{xz} + x(2y + x) e^{x/y}}{x y^2 e^{xz}} \text{ and}$$
- $$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x e^y + e^{xz} + x^2 \left(-\frac{x}{y^2}\right) e^{x/y}}{x y e^{xz}} = \frac{x^3 e^{x/y} - x y^2 e^y - y^2 e^{xz}}{x y^3 e^{xz}}.$$

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3. Here $\mathbf{u} = \cos \frac{\pi}{2} \mathbf{i} + \sin \frac{\pi}{2} \mathbf{j} = \mathbf{j}$, so $D_{\mathbf{u}}f(3, 0) = \frac{\partial f}{\partial y}(3, 0) = (x+1)e^y|_{(3,0)} = 4$.

5. $f_x(x, y) = \frac{\partial}{\partial x}(2x + 3xy - 3y + 4) = 2 + 3y$ and $f_y(x, y) = 3x - 3$, so
 $\nabla f(2, 1) = [(2 + 3y)\mathbf{i} + 3(x - 1)\mathbf{j}]_{(2,1)} = 5\mathbf{i} + 3\mathbf{j}$.

7. $f_x(x, y) = \frac{\partial}{\partial x}(x \sin y + y \cos x) = \sin y - y \sin x$ and $f_y(x, y) = x \cos y + \cos x$, so
 $\nabla f\left(\frac{\pi}{4}, \frac{\pi}{2}\right) = [(\sin y - y \sin x)\mathbf{i} + (x \cos y + \cos x)\mathbf{j}]_{(\pi/4, \pi/2)} = \left[1 - \frac{\pi}{2} \left(\frac{\sqrt{2}}{2}\right)\right]\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} = \frac{4 - \sqrt{2}\pi}{4}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$.

9. $f_x(x, y, z) = \frac{\partial}{\partial x}(xe^{yz}) = e^{yz}$, $f_y(x, y, z) = xze^{yz}$, and $f_z(x, y, z) = xye^{yz}$, so
 $\nabla f(1, 0, 2) = (e^{yz}\mathbf{i} + xze^{yz}\mathbf{j} + xye^{yz}\mathbf{k})|_{(1,0,2)} = \mathbf{i} + 2\mathbf{j}$.

11. Here $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} - 2\mathbf{j}}{\sqrt{1 + (-2)^2}} = \frac{\sqrt{5}}{5}\mathbf{i} - \frac{2\sqrt{5}}{5}\mathbf{j}$, $f_x(x, y) = 3x^2 - 2xy^2 + y$, and $f_y(x, y) = -2x^2y + x + 2y$, so

$$D_{\mathbf{u}}f(1, -1) = f_x(1, -1)u_1 + f_y(1, -1)u_2 \\ = \left[3(1)^2 - 2(1)(-1)^2 + (-1)\right]\left(\frac{\sqrt{5}}{5}\right) + \left[-2(1)^2(-1) + 1 + 2(-1)\right]\left(-\frac{2\sqrt{5}}{5}\right) = -\frac{2\sqrt{5}}{5}$$

15. Here $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-\mathbf{i} + 3\mathbf{j}}{\sqrt{(-1)^2 + 3^2}} = -\frac{\sqrt{10}}{10}\mathbf{i} + \frac{3\sqrt{10}}{10}\mathbf{j}$, $f_x(x, y) = -\frac{2y}{(x-y)^2}$, and $f_y(x, y) = \frac{2x}{(x-y)^2}$, so

$$D_{\mathbf{u}}f(2, 1) = f_x(2, 1)\left(-\frac{\sqrt{10}}{10}\right) + f_y(2, 1)\left(\frac{3\sqrt{10}}{10}\right) = -2\left(-\frac{\sqrt{10}}{10}\right) + 4\left(\frac{3\sqrt{10}}{10}\right) = \frac{7\sqrt{10}}{5}$$

29. Here $\mathbf{u} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{\mathbf{i} + 3\mathbf{j}}{\sqrt{1^2 + 3^2}} = \frac{\sqrt{10}}{10}\mathbf{i} + \frac{3\sqrt{10}}{10}\mathbf{j}$, $f_x(x, y) = 3x^2$, and $f_y(x, y) = 3y^2$, so

$$D_{\mathbf{u}}f(1, 2) = f_x(1, 2)\left(\frac{\sqrt{10}}{10}\right) + f_y(1, 2)\left(\frac{3\sqrt{10}}{10}\right) = 3\left(\frac{\sqrt{10}}{10}\right) + 12\left(\frac{3\sqrt{10}}{10}\right) = \frac{39\sqrt{10}}{10}$$

30. Here $\mathbf{u} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{-3\mathbf{i} + 2\mathbf{j}}{\sqrt{(-3)^2 + 2^2}} = -\frac{3\sqrt{13}}{13}\mathbf{i} + \frac{2\sqrt{13}}{13}\mathbf{j}$, $f_x(x, y) = e^{-y}$, and $f_y(x, y) = -xe^{-y}$, so

$$D_{\mathbf{u}}f(2, 0) = f_x(2, 0)\left(-\frac{3\sqrt{13}}{13}\right) + f_y(2, 0)\left(\frac{2\sqrt{13}}{13}\right) = 1\left(-\frac{3\sqrt{13}}{13}\right) - 2\left(\frac{2\sqrt{13}}{13}\right) = -\frac{7\sqrt{13}}{13}$$

31. Here $\mathbf{u} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{2\mathbf{i} + \frac{\pi}{4}\mathbf{j} - \frac{\pi}{6}\mathbf{k}}{\sqrt{2^2 + (\frac{\pi}{4})^2 + (\frac{\pi}{6})^2}} = \frac{12}{\sqrt{576 + 13\pi^2}} (2\mathbf{i} + \frac{\pi}{4}\mathbf{j} - \frac{\pi}{6}\mathbf{k})$, $f_x(x, y, z) = \sin(2y + 3z)$,

$f_y(x, y, z) = 2x \cos(2y + 3z)$, and $f_z(x, y, z) = 3x \cos(2y + 3z)$, so

$$D_{\mathbf{u}}f\left(1, \frac{\pi}{4}, -\frac{\pi}{12}\right) = f_x\left(1, \frac{\pi}{4}, -\frac{\pi}{12}\right) \cdot \frac{24}{\sqrt{576 + 13\pi^2}} + f_y\left(1, \frac{\pi}{4}, -\frac{\pi}{12}\right) \cdot \frac{3\pi}{\sqrt{576 + 13\pi^2}} + f_z\left(1, \frac{\pi}{4}, -\frac{\pi}{12}\right) \cdot \left(-\frac{2\pi}{\sqrt{576 + 13\pi^2}}\right)$$

$$= \frac{\sqrt{2}}{2} \left(\frac{24}{\sqrt{576 + 13\pi^2}}\right) + \sqrt{2} \left(\frac{3\pi}{\sqrt{576 + 13\pi^2}}\right) + \frac{3\sqrt{2}}{2} \left(-\frac{2\pi}{\sqrt{576 + 13\pi^2}}\right) = \frac{12\sqrt{2}}{\sqrt{13\pi^2 + 576}}$$

32. Here $\mathbf{u} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{\mathbf{i} + \mathbf{j} - 3\mathbf{k}}{\sqrt{1+1+9}} = \frac{\sqrt{11}}{11} (\mathbf{i} + \mathbf{j} - 3\mathbf{k})$, $f_x(x, y, z) = \frac{1}{y+z}$, $f_y(x, y, z) = \frac{z-x}{(y+z)^2}$, and

$f_z(x, y, z) = -\frac{x+y}{(y+z)^2}$, so

$$D_{\mathbf{u}}f(2, 1, 1) = f_x(2, 1, 1) \left(\frac{\sqrt{11}}{11}\right) + f_y(2, 1, 1) \left(\frac{\sqrt{11}}{11}\right) + f_z(2, 1, 1) \left(-\frac{3\sqrt{11}}{11}\right) = \frac{1}{2} \left(\frac{\sqrt{11}}{11}\right) - \frac{1}{4} \left(\frac{\sqrt{11}}{11}\right) - \frac{3}{4} \left(-\frac{3\sqrt{11}}{11}\right)$$

$$= \frac{5\sqrt{11}}{22}$$