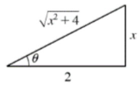
7.3 Trigonometric Substitutions

7. $\int \frac{dx}{x^2\sqrt{x^2+4}}$. Let $x = 2\tan\theta$, so $dx = 2\sec^2\theta\, d\theta$ and



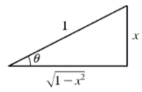
$\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4} = 2\sqrt{\tan^2\theta + 1} = 2\sec\theta$. Then

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta\, d\theta}{(4\tan^2\theta)(2\sec\theta)} = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta = \frac{1}{4}\int (\sin\theta)^{-2}\cos\theta\, d\theta$$

$$= -\frac{1}{4\sin\theta} + C = -\frac{\sqrt{x^2+4}}{4x} + C$$

9. $\int x^3\sqrt{1-x^2}\,dx$. Let $x = \sin\theta$, so $dx = \cos\theta\, d\theta$ and $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$.

Then



$$\int x^3\sqrt{1-x^2}\,dx = \int \sin^3\theta\cos\theta\cos\theta\, d\theta = \int \sin^2\theta\cos^2\theta\sin\theta\, d\theta$$

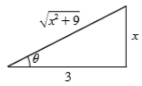
$$= \int \left(1-\cos^2\theta\right)\cos^2\theta\sin\theta\, d\theta = \int \left(\cos^2\theta - \cos^4\theta\right)\sin\theta\, d\theta$$

$$= -\tfrac{1}{3}\cos^3\theta + \tfrac{1}{5}\cos^5\theta + C = -\tfrac{1}{15}\left(\cos^3\theta\right)\left(5 - 3\cos^2\theta\right) + C$$

$$= -\tfrac{1}{15}\left(1-x^2\right)^{3/2}\left[5 - 3\left(1-x^2\right)\right] + C = -\tfrac{1}{15}\left(3x^2+2\right)\left(1-x^2\right)^{3/2} + C$$

Note that the problem can also be solved using the substitution $u = 1 - x^2$.

11. $\int \frac{x^3\,dx}{\sqrt{x^2+9}}$. Let $x = 3\tan\theta$, so $dx = 3\sec^2\theta\, d\theta$ and



$\sqrt{x^2+9} = \sqrt{9\tan^2\theta + 9} = 3\sqrt{\tan^2\theta+1} = 3\sec^2\theta$. Then

$$\int \frac{x^3\,dx}{\sqrt{x^2+9}} = \int \frac{(3\tan\theta)^3\left(3\sec^2\theta\, d\theta\right)}{3\sec\theta} = 27\int \tan^3\theta\sec\theta\, d\theta$$

$$= 27\int \tan^2\theta\sec\theta\tan\theta\, d\theta = 27\int\left(\sec^2\theta - 1\right)\sec\theta\tan\theta\, d\theta = 27\left(\tfrac{1}{3}\sec^3\theta - \sec\theta\right) + C$$

$$= 9\sec\theta\left(\sec^2\theta - 3\right) + C = 9\frac{\sqrt{x^2+9}}{3}\left(\frac{x^2+9}{9} - 3\right) + C = \tfrac{1}{3}\left(x^2-18\right)\sqrt{x^2+9} + C$$

Note that the problem can also be solved using the substitution $u = x^2 + 9$.

12. $I = \int_0^{3/4} \frac{x^2\,dx}{\sqrt{9-4x^2}}$. Let $u = 2x$, so $du = 2\,dx$, $x = 0 \Rightarrow u = 0$, and $x = \frac{3}{4} \Rightarrow u = \frac{3}{2}$. Then

$\int_0^{3/4} \frac{x^2\,dx}{\sqrt{9-4x^2}} = \int_0^{3/2} \frac{\left(\frac{1}{2}u\right)^2\left(\frac{1}{2}\,du\right)}{\sqrt{9-u^2}} = \frac{1}{8}\int_0^{3/2}\frac{u^2\,du}{\sqrt{9-u^2}}$. Now let $u = 3\sin\theta$, so $du = 3\cos\theta\, d\theta$. Then

$\sqrt{9-u^2} = \sqrt{9-9\sin^2\theta} = 3\sqrt{1-\sin^2\theta} = 3\sqrt{\cos^2\theta} = 3\cos\theta$, $u = 0 \Rightarrow \theta = 0$, and $u = \frac{3}{2} \Rightarrow 3\sin\theta = \frac{3}{2} \Rightarrow \theta = \frac{\pi}{6}$.

Thus,

$$I = \frac{1}{8}\int_0^{3/2}\frac{u^2\,du}{\sqrt{9-u^2}} = \frac{1}{8}\int_0^{\pi/6}\frac{(3\sin\theta)^2(3\cos\theta\, d\theta)}{3\cos\theta} = \frac{9}{8}\int_0^{\pi/6}\sin^2\theta\, d\theta = \frac{9}{8}\int_0^{\pi/6}\frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \tfrac{9}{16}\left(\theta - \tfrac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/6} = \tfrac{9}{16}\left(\tfrac{\pi}{6} - \tfrac{\sqrt{3}}{4}\right) = \tfrac{3}{64}\left(2\pi - 3\sqrt{3}\right)$$

7.4 The Method of Partial Fractions

7. $I = \int \frac{dx}{x(x-4)}$. Now $\frac{1}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4} = \frac{(A+B)x - 4A}{x(x-4)} \Rightarrow A + B = 0$ and $-4A = 1 \Rightarrow A = -\frac{1}{4}$ and

$B = \frac{1}{4}$, so $I = -\frac{1}{4}\int\frac{dx}{x} + \frac{1}{4}\int\frac{dx}{x-4} = -\frac{1}{4}\ln|x| + \frac{1}{4}\ln|x-4| + C = \frac{1}{4}\ln\left|\frac{x-4}{x}\right| + C$.

21. $I = \int \frac{4x^2+3x+2}{x^3+x^2}\,dx$. Now

$$\frac{4x^2+3x+2}{x^3+x^2} = \frac{4x^2+3x+2}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} = \frac{Ax(x+1) + B(x+1) + Cx^2}{x^2(x+1)}$$

$$= \frac{A\left(x^2+x\right) + B(x+1) + Cx^2}{x^2(x+1)} = \frac{(A+C)x^2 + (A+B)x + B}{x^2(x+1)}$$

$\Rightarrow A + C = 4$, $A + B = 3$, and $B = 2$, leading to $A = 1$ and $C = 3$. Thus,

$$I = \int\left(\frac{1}{x} + \frac{2}{x^2} + \frac{3}{x+1}\right)dx = \ln|x| - \frac{2}{x} + 3\ln|x+1| + C = \ln\left|x(x+1)^3\right| - \frac{2}{x} + C.$$

31. $I = \int \frac{x^3+3}{(x+1)(x^2+1)}\,dx = \int \left[1 - \frac{x^2+x-2}{(x+1)(x^2+1)}\right] dx = x - J,$

$$\begin{array}{r|l} & 1 \\ x^3+x^2+x+1 & x^3 \qquad\quad +3 \\ & \underline{x^3+x^2+x+1} \\ & -x^2-x+2 \end{array}$$

where $J = \int \frac{x^2+x-2}{(x+1)(x^2+1)}\,dx$. Now

$$\frac{x^2+x-2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{(A+B)x^2+(B+C)x+(A+C)}{(x+1)(x^2+1)}$$

so $A+B=1$, $B+C=1$, and $A+C=-2$. Solving, we obtain $A=-1$, $B=2$, and $C=-1$,

so $J = -\int \frac{dx}{x+1} + 2\int \frac{x\,dx}{x^2+1} - \int \frac{dx}{x^2+1} = -\ln|x+1| + \ln\left(x^2+1\right) - \tan^{-1}x + C_1$. Finally,

$I = x + \tan^{-1}x + \ln\left|\frac{x+1}{x^2+1}\right| + C.$

33. $I = \int \frac{5x^3-3x^2+7x-3}{\left(x^2+1\right)^2}\,dx$. Now

$\frac{5x^3-3x^2+7x-3}{\left(x^2+1\right)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{\left(x^2+1\right)^2} = \frac{Ax^3+Bx^2+(A+C)x+B+D}{\left(x^2+1\right)^2}$, so $A=5$,

$B=-3$, $A+C=7$, and $B+D=-3$. Solving, we obtain $A=5$, $B=-3$, $C=2$, and $D=0$, so

$I = \int \frac{5x\,dx}{x^2+1} - \int \frac{3\,dx}{x^2+1} + \int \frac{2x\,dx}{\left(x^2+1\right)^2} = \frac{5}{2}\ln\left(x^2+1\right) - 3\tan^{-1}x - \frac{1}{x^2+1} + C.$

7.6 Improper Integrals

7. $\int_1^\infty \frac{dx}{x^3} = \lim_{b\to\infty}\int_1^b x^{-3}\,dx = \lim_{b\to\infty}\left[-\frac{1}{2x^2}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{2b^2}+\frac{1}{2}\right) = \frac{1}{2}$

17. $\int_0^\infty \frac{x\,dx}{1+x^2} = \lim_{b\to\infty}\int_0^b \frac{x\,dx}{1+x^2} = \lim_{b\to\infty}\left[\frac{1}{2}\ln\left(1+x^2\right)\right]_0^b = \lim_{b\to\infty}\left[\frac{1}{2}\ln\left(1+b^2\right)\right] = \infty$, so the integral diverges.

23.
$$\begin{aligned}\int_{-\infty}^\infty \frac{x\,dx}{\left(x^2+1\right)^{3/2}} &= \int_{-\infty}^0 \frac{x\,dx}{\left(x^2+1\right)^{3/2}} + \int_0^\infty \frac{x\,dx}{\left(x^2+1\right)^{3/2}} = \lim_{a\to-\infty}\int_a^0 \frac{x\,dx}{\left(x^2+1\right)^{3/2}} + \lim_{b\to\infty}\int_0^b \frac{x\,dx}{\left(x^2+1\right)^{3/2}} \\ &= \lim_{a\to-\infty}\left[\tfrac{1}{2}(-2)\left(x^2+1\right)^{-\frac{1}{2}}\right]_a^0 + \lim_{b\to\infty}\left[\tfrac{1}{2}(-2)\left(x^2+1\right)^{-\frac{1}{2}}\right]_0^b \\ &= \lim_{a\to-\infty}\left(-1+\frac{1}{\sqrt{a^2+1}}\right) + \lim_{b\to\infty}\left(-\frac{1}{\sqrt{b^2+1}}+1\right) = -1+1 = 0\end{aligned}$$

41. Working with the indefinite integral $I = \int \frac{\ln x}{\sqrt{x}}\,dx$, we use parts with $u = \ln x$ and $dv = x^{-1/2}\,dx \Rightarrow$

$du = dx/x$ and $v = 2\sqrt{x}$: $I = 2\sqrt{x}\ln x - \int 2x^{-1/2}\,dx = 2\sqrt{x}\ln x - 4\sqrt{x} + C = 2\sqrt{x}(\ln x - 2) + C$,

so $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{a\to0^+}\left[2\sqrt{x}(\ln x-2)\right]_a^1 = \lim_{a\to0^+}\left[-4-2\sqrt{a}(\ln a-2)\right] = -4$ since, by l'Hôpital's Rule,

$\lim_{a\to0^+}\frac{\ln a}{a^{-1/2}} = \lim_{a\to0^+}\frac{1/a}{-\frac{1}{2}a^{-3/2}} = \lim_{a\to0^+}\left(-2\sqrt{a}\right) = 0.$