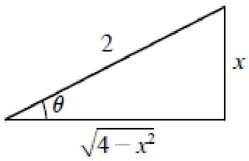
7. 3

3. $\int x\sqrt{4-x^2}\,dx$. Let $x = 2\sin\theta$, so $dx = 2\cos\theta\,d\theta$ and

$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\cos\theta$. Then



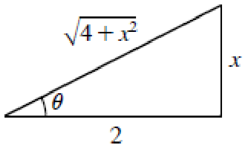
$$\int x\sqrt{4-x^2}\,dx = \int (2\sin\theta)(2\cos\theta)(2\cos\theta)\,d\theta = 8\int \cos^2\theta\sin\theta\,d\theta$$

$$= -\frac{8\cos^3\theta}{3} + C = -\frac{8}{3}\left(\frac{\sqrt{4-x^2}}{2}\right)^3 + C = -\frac{\left(4-x^2\right)^{3/2}}{3} + C$$

Note that the problem can be solved more easily using the substitution $u = 4 - x^2$.

5. $\int \frac{dx}{x\sqrt{4+x^2}}$. Let $x = 2\tan\theta$, so $dx = 2\sec^2\theta\,d\theta$ and

$\sqrt{4+x^2} = \sqrt{4+4\tan^2\theta} = 2\sqrt{1+\tan^2\theta} = 2\sec\theta$. Then



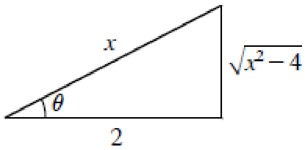
$$\int \frac{dx}{x\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\,d\theta}{(2\tan\theta)(2\sec\theta)} = \frac{1}{2}\int \csc\theta\,d\theta = \frac{1}{2}\int \frac{\csc\theta\,(\csc\theta+\cot\theta)}{\csc\theta+\cot\theta}\,d\theta$$

$$= \frac{1}{2}\int \frac{\csc^2\theta + \csc\theta\cot\theta}{\csc\theta+\cot\theta}\,d\theta = -\frac{1}{2}\ln\left|\csc\theta+\cot\theta\right| + C$$

$$= -\frac{1}{2}\ln\left|\frac{\sqrt{4+x^2}}{x} + \frac{2}{x}\right| + C = -\frac{1}{2}\ln\left|\frac{\sqrt{4+x^2}+2}{x}\right| + C$$

8. $\int \frac{dx}{x^3\sqrt{x^2-4}}$. Let $x = 2\sec\theta$, so $dx = 2\sec\theta\tan\theta\,d\theta$ and

$\sqrt{x^2-4} = \sqrt{4\sec^2\theta - 4} = 2\sqrt{\sec^2\theta - 1} = 2\tan\theta$. Then



$$\int \frac{dx}{x^3\sqrt{x^2-4}} = \int \frac{2\sec\theta\tan\theta\,d\theta}{(8\sec^3\theta)(2\tan\theta)} = \frac{1}{8}\int \cos^2\theta\,d\theta = \frac{1}{8}\int \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \tfrac{1}{16}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{1}{16}\left(\theta + \sin\theta\cos\theta\right) + C$$

$$= \frac{1}{16}\left(\sec^{-1}\frac{x}{2} + \frac{\sqrt{x^2-4}}{x}\cdot\frac{2}{x}\right) + C$$

$$= \frac{1}{16}\left(\sec^{-1}\frac{x}{2} + \frac{2\sqrt{x^2-4}}{x^2}\right) + C$$

21. $I = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4-x^2}\,dx = 2\int_0^{\sqrt{3}} \sqrt{4-x^2}\,dx$. Let $x = 2\sin\theta$, so $dx = 2\cos\theta\,d\theta$, $\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\cos\theta$,

$x = 0 \Rightarrow \theta = 0$, and $x = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$. Then

$$I = 2\int_0^{\pi/3} (2\cos\theta)(2\cos\theta)\,d\theta = 8\int_0^{\pi/3} \cos^2\theta\,d\theta = 4\int_0^{\pi/3} (1+\cos 2\theta)\,d\theta = 4\left(\theta + \tfrac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/3}$$

$$= 4\left(\tfrac{\pi}{3} + \tfrac{\sqrt{3}}{4}\right) = \tfrac{1}{3}\left(4\pi + 3\sqrt{3}\right)$$

25. $I = \int e^x \sqrt{4 - e^{2x}}\, dx$. Let $u = e^x$, so $du = e^x\, dx$. Then $I = \int \sqrt{4 - u^2}\, du$. Next, let $u = 2\sin\theta$, so $du = 2\cos\theta\, d\theta$ and $\sqrt{4 - u^2} = \sqrt{4 - 4\sin^2\theta} = 2\sqrt{1 - \sin^2\theta} = 2\cos\theta$. Then

$$I = \int (2\cos\theta)(2\cos\theta\, d\theta) = 4\int \cos^2\theta\, d\theta = 4\int \frac{1 + \cos 2\theta}{2}\, d\theta = 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C$$

$$= 2\left[\sin^{-1}\left(\tfrac{1}{2}u\right) + \tfrac{1}{2}u\left(\tfrac{1}{2}\sqrt{4 - u^2}\right)\right] + C$$

Therefore, $\int e^x \sqrt{4 - e^{2x}}\, dx = 2\sin^{-1}\left(\tfrac{1}{2}e^x\right) + \tfrac{1}{2}e^x\sqrt{4 - e^{2x}} + C$.

7. 4

7. $I = \int \frac{dx}{x\,(x-4)}$. Now $\frac{1}{x\,(x-4)} = \frac{A}{x} + \frac{B}{x-4} = \frac{(A+B)\,x - 4A}{x\,(x-4)} \Rightarrow A + B = 0$ and $-4A = 1 \Rightarrow A = -\frac{1}{4}$ and $B = \frac{1}{4}$, so $I = -\frac{1}{4}\int \frac{dx}{x} + \frac{1}{4}\int \frac{dx}{x-4} = -\frac{1}{4}\ln|x| + \frac{1}{4}\ln|x-4| + C = \frac{1}{4}\ln\left|\frac{x-4}{x}\right| + C.$

11. $I = \int_3^4 \frac{dx}{x^2-4}$. Now $\frac{1}{x^2-4} = \frac{1}{(x+2)\,(x-2)} = \frac{A}{x+2} + \frac{B}{x-2} = \frac{(A+B)\,x - 2\,(A-B)}{(x+2)\,(x-2)} \Rightarrow A + B = 0$ and $-2A + 2B = 1$. The first equation gives $B = -A$, and substituting into the second, we find $-4A = 1 \Rightarrow A = -\frac{1}{4}$, so $B = \frac{1}{4}$. Then

$$I = \int_3^4 \left(\frac{-\frac{1}{4}}{x+2} + \frac{\frac{1}{4}}{x-2}\right)dx = \left(-\frac{1}{4}\ln|x+2| + \frac{1}{4}\ln|x-2|\right)\Big|_3^4 = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|\Big|_3^4$$

$$= \frac{1}{4}\left(\ln\frac{2}{6} - \ln\frac{1}{5}\right) = \frac{1}{4}\left(\ln 2 - \ln 6 - \ln 1 + \ln 5\right) = \frac{1}{4}\ln\frac{5}{3}$$

21. $I = \int \frac{4x^2+3x+2}{x^3+x^2}\,dx$. Now

$$\frac{4x^2+3x+2}{x^3+x^2} = \frac{4x^2+3x+2}{x^2\,(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} = \frac{Ax\,(x+1) + B\,(x+1) + Cx^2}{x^2\,(x+1)}$$

$$= \frac{A\left(x^2+x\right) + B\,(x+1) + Cx^2}{x^2\,(x+1)} = \frac{(A+C)\,x^2 + (A+B)\,x + B}{x^2\,(x+1)}$$

$\Rightarrow A + C = 4$, $A + B = 3$, and $B = 2$, leading to $A = 1$ and $C = 3$. Thus,

$$I = \int \left(\frac{1}{x} + \frac{2}{x^2} + \frac{3}{x+1}\right)dx = \ln|x| - \frac{2}{x} + 3\ln|x+1| + C = \ln\left|x\,(x+1)^3\right| - \frac{2}{x} + C.$$

31. $I = \int \frac{x^3+3}{(x+1)\left(x^2+1\right)}\,dx = \int \left[1 - \frac{x^2+x-2}{(x+1)\left(x^2+1\right)}\right]dx = x - J,$

$$\begin{array}{r|l} & \quad 1 \\ x^3+x^2+x+1 & x^3 \qquad\qquad +3 \\ & x^3 + x^2 + x + 1 \\ \hline & \quad -x^2 - x + 2 \end{array}$$

where $J = \int \frac{x^2+x-2}{(x+1)\left(x^2+1\right)}\,dx$. Now

$$\frac{x^2+x-2}{(x+1)\left(x^2+1\right)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$= \frac{(A+B)\,x^2 + (B+C)\,x + (A+C)}{(x+1)\left(x^2+1\right)}$$

so $A + B = 1$, $B + C = 1$, and $A + C = -2$. Solving, we obtain $A = -1$, $B = 2$, and $C = -1$, so $J = -\int \frac{dx}{x+1} + 2\int \frac{x\,dx}{x^2+1} - \int \frac{dx}{x^2+1} = -\ln|x+1| + \ln\left(x^2+1\right) - \tan^{-1}x + C_1$. Finally,

$$I = x + \tan^{-1}x + \ln\left|\frac{x+1}{x^2+1}\right| + C.$$

40. $I = \int \frac{\left(3x - x^2\right) dx}{\left(x^2+1\right)\left(x^2+2\right)}$. Now

$\frac{\left(3x - x^2\right) dx}{\left(x^2+1\right)\left(x^2+2\right)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2} = \frac{(A+C)\,x^3 + (B+D)\,x^2 + (2A+C)\,x + 2B + D}{\left(x^2+1\right)\left(x^2+2\right)}$, so $A + C = 0$, $B + D = -1$, $2A + C = 3$, and $2B + D = 0$. Solving, we find $A = 3$, $B = 1$, $C = -3$, and $D = -2$, so

$$I = \int \left(\frac{3x}{x^2+1} + \frac{1}{x^2+1} - \frac{3x}{x^2+2} - \frac{2}{x^2+2} \right) dx = \tfrac{3}{2} \ln\left(x^2+1\right) + \tan^{-1} x - \tfrac{3}{2} \ln\left(x^2+2\right) - \sqrt{2}\tan^{-1} \tfrac{\sqrt{2}}{2} x + C$$

$$= \tfrac{3}{2} \ln \frac{x^2+1}{x^2+2} + \tan^{-1} x - \sqrt{2}\tan^{-1} \tfrac{\sqrt{2}}{2} x + C$$

7. 6

11. $\int_1^\infty \frac{dx}{(x+2)^{3/2}} = \lim_{b\to\infty} \int_1^b (x+2)^{-3/2}\,dx = \lim_{b\to\infty}\left[\frac{-2}{(x+2)^{1/2}}\right]_1^b = \lim_{b\to\infty}\left[\frac{-2}{(b+2)^{1/2}} + \frac{2\sqrt{3}}{3}\right] = \frac{2\sqrt{3}}{3}$

15. $\int_0^\infty \sin x\,dx = \lim_{b\to\infty}\int_0^b \sin x\,dx = \lim_{b\to\infty}[-\cos x]_0^b = \lim_{b\to\infty}(-\cos b + 1)$, which does not exist. The integral diverges.

19. $\int_{-\infty}^\infty \frac{dx}{x^2+4} = \int_{-\infty}^0 \frac{dx}{x^2+4} + \int_0^\infty \frac{dx}{x^2+4} = 2\int_0^\infty \frac{dx}{x^2+4} = 2\lim_{b\to\infty}\int_0^b \frac{dx}{x^2+4} = 2\cdot\frac{1}{2}\lim_{b\to\infty}\tan^{-1}\left(\frac{1}{2}x\right)\Big|_0^b$

$= \lim_{b\to\infty}\left[\tan^{-1}\left(\frac{1}{2}b\right) - 0\right] = \frac{\pi}{2}$

41. Working with the indefinite integral $I = \int \frac{\ln x}{\sqrt{x}}\,dx$, we use parts with $u = \ln x$ and $dv = x^{-1/2}\,dx \Rightarrow$ $du = dx/x$ and $v = 2\sqrt{x}$: $I = 2\sqrt{x}\ln x - \int 2x^{-1/2}\,dx = 2\sqrt{x}\ln x - 4\sqrt{x} + C = 2\sqrt{x}(\ln x - 2) + C$, so $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\left[2\sqrt{x}(\ln x - 2)\right]_a^1 = \lim_{a\to 0^+}\left[-4 - 2\sqrt{a}(\ln a - 2)\right] = -4$ since, by l'Hôpital's Rule, $\lim_{a\to 0^+}\frac{\ln a}{a^{-1/2}} = \lim_{a\to 0^+}\frac{1/a}{-\frac{1}{2}a^{-3/2}} = \lim_{a\to 0^+}\left(-2\sqrt{a}\right) = 0.$

68. a. Let $u = \frac{1}{x}$, so $du = -\frac{dx}{x^2}$, $x \to 0^+ \Rightarrow u \to \infty$, and $x \to \infty \Rightarrow u \to 0$. Then

$I = \int_0^\infty \frac{x^2\,dx}{x^4+1} = -\int_\infty^0 \frac{\frac{1}{u^4}\,du}{\frac{1}{u^4}+1} = \int_0^\infty \frac{du}{1+u^4} = \int_0^\infty \frac{dx}{x^4+1}$. Adding I to both sides of the last equation gives

$2I = \int_0^\infty \frac{x^2\,dx}{x^4+1} + \int_0^\infty \frac{dx}{x^4+1} = \int_0^\infty \frac{x^2+1}{x^4+1}\,dx \Rightarrow I = \frac{1}{2}\int_0^\infty \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx.$

b. Let $v = x - \frac{1}{x}$, so $dv = \left(1 + \frac{1}{x^2}\right)dx$, $x \to 0^+ \Rightarrow v \to -\infty$, and $x \to \infty \Rightarrow v \to \infty$. Also,

$v^2 + 2 = \left(x - \frac{1}{x}\right)^2 + 2 = x^2 - 2 + \frac{1}{x^2} + 2 = x^2 + \frac{1}{x^2}$. Thus, $I = \frac{1}{2}\int_{-\infty}^\infty \frac{dv}{v^2+2}$.

c. $I = \int_0^\infty \frac{x^2\,dx}{x^4+1} = \frac{1}{2}\int_{-\infty}^\infty \frac{dv}{v^2+2} = \int_0^\infty \frac{dv}{v^2+2}$ because the integrand is even. Continuing,

$I = \lim_{b\to\infty}\int_0^b \frac{dv}{v^2+2} = \lim_{b\to\infty}\left(\frac{1}{\sqrt{2}}\tan^{-1}\frac{v}{\sqrt{2}}\right)\Big|_0^b = \frac{\sqrt{2}}{2}\lim_{b\to\infty}\left(\tan^{-1}b - 0\right) = \frac{\sqrt{2}}{2}\cdot\frac{\pi}{2} = \frac{\sqrt{2}}{4}\pi.$