

## 9.7 Power Series

1. Let  $u = \frac{x^n}{n+1}$ . Then  $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{n+2}}{\frac{x^n}{n+1}} \right| = \lim_{n \rightarrow \infty} \left[ \left( \frac{n+1}{n+2} \right) |x| \right] = |x|$ , so the radius of convergence is 1.

The series converges for  $-1 < x < 1$ . At  $x = -1$  the series is  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ , which converges, and at  $x = 1$  it is  $\sum_{n=0}^{\infty} \frac{1}{n+1}$ , which diverges. Thus, the interval of convergence is  $[-1, 1)$ .

5. Let  $u_n = \frac{(2x)^n}{n!}$ . Then  $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2x)^{n+1}}{(n+1)!} \cdot \frac{n!}{(2x)^n} \right| = \lim_{n \rightarrow \infty} \left( \frac{2}{n+1} \right) |x| = 0$  for any real  $x$ , so the radius of convergence is infinite and the interval of convergence is  $(-\infty, \infty)$ .

7. Let  $u_n = (nx)^n = n^n x^n$ . Then  $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^{n+1} x^{n+1}}{n^n x^n} \right| = \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^n (n+1) |x| = \infty$  for  $x \neq 0$ , so the radius of convergence is 0 and the series converges only at  $x = 0$ .

9. Let  $u_n = \frac{x^n}{\ln n}$ . Then  $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{x^n} \right| = \lim_{n \rightarrow \infty} \frac{\ln n}{\ln(n+1)} \cdot |x| = |x|$ , so the radius of convergence is  $R = 1$  and the series converges for  $-1 < x < 1$ . At  $x = -1$  the series is  $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ , which converges, and at  $x = 1$  it is  $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ , which diverges. Thus, the interval of convergence is  $[-1, 1)$ .

11. Let  $u_n = \frac{e^n x^n}{n}$ . Then  $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{e^{n+1} x^{n+1}}{n+1} \cdot \frac{n}{e^n x^n} \right| = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right) e |x| = e |x|$ , so the radius of convergence is  $\frac{1}{e}$  and the series converges for  $-\frac{1}{e} < x < \frac{1}{e}$ . At  $x = -\frac{1}{e}$ , the series is  $\sum \frac{(-1)^n}{n}$ , which converges, and at  $x = \frac{1}{e}$  it is  $\sum \frac{1}{n}$ , which diverges. Thus, the interval of convergence is  $\left[ -\frac{1}{e}, \frac{1}{e} \right)$ .

12. Let  $u_n = \frac{(-1)^n n! x^n}{2^n}$ . Then  $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} (n+1)! x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(-1)^n n! x^n} \right| = \lim_{n \rightarrow \infty} \left( \frac{n+1}{2} \right) |x| = \infty$  for  $x \neq 0$ , so  $R = 0$  and the series converges only at  $x = 0$ .

15. Let  $u_n = \frac{(-1)^{n-1} (x-2)^n}{n \cdot 3^n}$ . Then

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^n (x-2)^{n+1}}{(n+1) 3^{n+1}} \cdot \frac{n \cdot 3^n}{(-1)^{n-1} (x-2)^n} \right| = \lim_{n \rightarrow \infty} \frac{1}{3} \left( \frac{n}{n+1} \right) |x-2| = \frac{1}{3} |x-2| < 1$$

$\Leftrightarrow |x-2| < 3$ , so the radius of convergence is 3 and the series converges on  $(-1, 5)$ . At  $x = -1$  the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} (-3)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n}, \text{ which diverges, and at } x = 5 \text{ it is } \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 3^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}, \text{ which}$$

converges. Thus, the interval of convergence is  $(-1, 5]$ .

19. Let  $u_n = \frac{(-1)^n (x+2)^{2n+1}}{(2n+1)!}$ . Then

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} (x+2)^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n (x+2)^{2n+1}} \right| = \lim_{n \rightarrow \infty} \frac{|x+2|^2}{(2n+2)(2n+3)} = 0 \text{ for any real } x, \text{ so}$$

$R = \infty$  and the interval of convergence is  $(-\infty, \infty)$ .

25. Let  $u_n = \frac{x^n}{n(\ln n)^2}$ . Then

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)[\ln(n+1)]^2} \cdot \frac{n(\ln n)^2}{x^n} \right| = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right) \left[ \frac{\ln n}{\ln(n+1)} \right]^2 |x| = |x|, \text{ so } R = 1 \text{ and the}$$

series converges on  $(-1, 1)$ . At  $x = -1$  the series is  $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$ , which converges, and at  $x = 1$  it is  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ , which also converges (see Exercise 9.3.33). Thus, the interval of convergence is  $[-1, 1]$ .

38.  $f(x) = \sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2 3^n}$ .  $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2 3^{n+1}} \cdot \frac{n^2 3^n}{(x-2)^n} \right| = \lim_{n \rightarrow \infty} \frac{1}{3} \left( \frac{n}{n+1} \right)^2 |x-2| = \frac{1}{3} |x-2| < 1$

$\Rightarrow |x-2| < 3$ , so the series converges on  $(-1, 5)$ . At  $x = -1$  the series is  $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ ,

which converges, and at  $x = 5$  it is  $\sum_{n=1}^{\infty} \frac{3^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ , which also converges. Therefore,

the interval of convergence is  $[-1, 5]$ .  $f'(x) = \sum_{n=1}^{\infty} \frac{n(x-2)^{n-1}}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{(x-2)^{n-1}}{n 3^n}$  and

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-2)^n}{(n+1) 3^{n+1}} \cdot \frac{n 3^n}{(x-2)^{n-1}} \right| = \lim_{n \rightarrow \infty} \frac{1}{3} \left( \frac{n}{n+1} \right) |x-2| = \frac{1}{3} |x-2| < 1 \Rightarrow |x-2| < 3 \Rightarrow f'$$

converges on  $(-1, 5)$ . At  $x = -1$  the series is  $f'(-1) = \sum_{n=1}^{\infty} \frac{n(-3)^{n-1}}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n}$ , which converges, and at  $x = 5$

it is  $f'(5) = \sum_{n=1}^{\infty} \frac{n 3^{n-1}}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{1}{3n}$ , which diverges. The interval of convergence is thus  $[-1, 5)$ .

$$39. f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} \Rightarrow f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n^2} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} \text{ and}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^n}{n+1} \cdot \frac{n}{x^{n-1}} \right| = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right) |x| = |x|, \text{ so } f' \text{ converges on } (-1, 1). \text{ If}$$

$$x = -1 \text{ the series is } f'(-1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}, \text{ which converges, and at } x = 1 \text{ it is } f'(1) = \sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges. Thus,  $f'$  has interval of convergence  $[-1, 1)$ .  $f''(x) = \sum_{n=2}^{\infty} \frac{(n-1)x^{n-2}}{n}$  and

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{nx^{n-1}}{n+1} \cdot \frac{n}{(n-1)x^{n-2}} \right| = \lim_{n \rightarrow \infty} \left( \frac{n^2}{n^2-1} \right) |x| = |x|, \text{ so } f'' \text{ converges on } (-1, 1). \text{ At } x = -1$$

we have  $f''(-1) = \sum_{n=2}^{\infty} \frac{(n-1)(-1)^{n-2}}{n}$ , and since  $\lim_{n \rightarrow \infty} \left( \frac{n-1}{n} \right) = 1$ ,  $\lim_{n \rightarrow \infty} \frac{(n-1)(-1)^{n-2}}{n}$  does not exist, and

the Divergence Test shows that  $f''(-1)$  is divergent. Similarly,  $f''(1) = \sum_{n=2}^{\infty} \left( \frac{n-1}{n} \right)$  is divergent, so the interval of

convergence of  $f''$  is  $(-1, 1)$ . It is easy to see that the interval of convergence of  $f$  is  $[-1, 1]$ .

$$40. \sum_{n=1}^{\infty} \frac{\sin(n^3 x)}{n^2}. \text{ Since } -\frac{1}{n^2} \leq \frac{\sin(n^3 x)}{n^2} \leq \frac{1}{n^2} \text{ and both } \sum_{n=1}^{\infty} \left( -\frac{1}{n^2} \right) \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converge, the Comparison}$$

Test implies that the given series converges for all real values of  $x$ .  $\sum_{n=1}^{\infty} \frac{d}{dx} \left[ \frac{\sin(n^3 x)}{n^2} \right] = \sum_{n=1}^{\infty} n \cos(n^3 x)$ . Since

$\lim_{n \rightarrow \infty} n \cos(n^3 x) \neq 0$ , the Divergence Theorem shows that this series is divergent. This does not contradict Theorem 2 because the given series is not a power series.

## 9.8 Taylor and Maclaurin Series

$$3. f(x) = e^x, f'(x) = e^x, f''(x) = e^x, \dots, f^{(n)}(x) = e^x, \dots$$

$$f(2) = e^2, f'(2) = e^2, f''(2) = e^2, \dots, f^{(n)}(2) = e^2, \dots$$

The required Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!} (x-2)^n = e^2 \left[ 1 + (x-2) + \frac{1}{2!} (x-2)^2 + \dots + \frac{1}{n!} (x-2)^n + \dots \right].$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{e^2 (x-2)^{n+1}}{(n+1)!} \cdot \frac{n!}{e^2 (x-2)^n} \right| = \lim_{n \rightarrow \infty} \left( \frac{1}{n+1} \right) |x-2| = 0, \text{ so } R = \infty.$$

$$5. f(x) = \sin 2x, f'(x) = 2 \cos 2x, f''(x) = (-1)^1 2^2 \sin 2x, f'''(x) = (-1)^1 2^3 \cos 2x, f^{(4)}(x) = (-1)^2 2^4 \sin 2x, \dots$$

$$f(0) = 0, f'(0) = 2, f''(0) = 0, f'''(0) = -2^3, f^{(4)}(0) = 0, \dots$$

$$\text{The required Maclaurin series is } \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 2x - \frac{2^3}{3!} x^3 + \frac{2^5}{5!} x^5 - \dots + \frac{(-1)^k 2^{2k+1}}{(2k+1)!} x^{2k+1} + \dots$$

Note that we have changed the index variable to  $k$ , where  $n = 2k + 1$ .

$$\lim_{k \rightarrow \infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(-1)^{k+1} 2^{2k+3} x^{2k+3}}{(2k+3)!} \cdot \frac{(2k+1)!}{(-1)^k (2)^{2k+1} x^{2k+1}} \right| = \lim_{k \rightarrow \infty} \frac{4}{(2k+2)(2k+3)} |x|^2 = 0, \text{ so } R = \infty.$$

$$9. f(x) = \ln x, f'(x) = \frac{1}{x}, f''(x) = -\frac{1}{x^2}, f'''(x) = \frac{2}{x^3}, \dots, f^{(n)}(x) = \frac{(-1)^{n-1} (n-1)!}{x^n}, \dots$$

$$f(2) = \ln 2, f'(2) = \frac{1}{2}, f''(2) = -\frac{1}{2^2}, f'''(2) = \frac{2}{2^3}, \dots, f^{(n)}(2) = \frac{(-1)^{n-1} (n-1)!}{2^n}, \dots$$

The required Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (n-1)!}{n! 2^n} (x-2)^n = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 2^n} (x-2)^n.$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^n (x-2)^{n+1}}{(n+1) 2^{n+1}} \cdot \frac{n 2^n}{(-1)^{n-1} (x-2)^n} \right| = \lim_{n \rightarrow \infty} \frac{1}{2} \left( \frac{n}{n+1} \right) |x-2| = \frac{1}{2} |x-2| < 1 \text{ or } |x-2| < 2, \text{ so } R = 2.$$

$$11. f(x) = \frac{1}{1+x} = \frac{1}{2+(x-1)} = \frac{1}{2 \left[ 1 + \left( \frac{x-1}{2} \right) \right]} = \frac{1}{2} \cdot \frac{1}{1 - \left[ -\left( \frac{x-1}{2} \right) \right]} = \frac{1}{2} \sum_{n=0}^{\infty} \left[ -\left( \frac{x-1}{2} \right) \right]^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (x-1)^n$$

The series converges for  $\frac{|x-1|}{2} < 1$  or  $|x-1| < 2$ , so  $R = 2$ .

$$17. f(x) = x e^{-x} = x \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n!} \text{ with } R = \infty.$$

$$19. f(x) = x^2 \cos x = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n)!} \text{ with } R = \infty.$$

$$27. f(x) = \ln(1+x^2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x^2)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{n} \text{ with } R = 1.$$

$$45. \frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} (-1)^n x^{3n}, \text{ so } \int \frac{dx}{1+x^3} = \int \sum_{n=0}^{\infty} (-1)^n x^{3n} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n+1} + C.$$

$$49. \frac{\ln(1+x)}{x} = \frac{1}{x} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \text{ so } \int \frac{\ln(1+x)}{x} dx = \int \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n-1}}{n} dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n^2} + C.$$