Calculus Exam-3 (107.06.26)

- **1. (15%) Show that** $\lim_{x,y\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ **does not exists.**

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◦ Proof :

The function is defined everywhere except at $(0,0)$. Let's approach $(0,0)$ along two different paths

$$\text{along x-axis} \to \lim_{x,y\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} = \frac{x^2}{x^2} = 1$$
$$\text{along y-axis} \to \lim_{x,y\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\to 0} = \frac{-y^2}{y^2} = -1$$

Since $1 \neq -1$, we conclude that the given limit does not exist.

- **2. (15%) Find the relative extrema of** $f(x,y) = x^3 + y^3 - 2xy + 7x - 8y + 2$

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◦ Sol :

$$f_x(x,y) = 3x^2 - 2y + 7 \quad ; \quad f_y(x,y) = 2y - 2x - 8$$

Solving $f_x(x,y) = 0$ and $f_y(x,y) = 0$, we have

$$\begin{cases} 3x^2 - 2y + 7 = 0 \\ 2y - 2x - 8 = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 5 \end{cases} \text{ or } \begin{cases} x = -1/3 \\ y = 11/3 \end{cases}$$

Hence the critical points are $(-1/3, 11/3)$ and $(1,5)$.

Next, we use Second Derivative Test

$$f_{xx}(x,y) = 6x \quad ; \quad f_{yy}(x,y) = 2 \quad ; \quad f_{xy}(x,y) = f_{yx}(x,y) = -2$$

$$\textit{then} \qquad D = f_{xx}f_{yy} - f_{xy}^2 = 12x - 4$$
$$D(1,5) > 0 \quad ; \quad D(-1/3, 11/3) < 0$$

Since $f_{xx}(1,5) > 0$, point $(1,5)$ gives the relative extrema minimum of f with value is $f(1,5) = -15$.
And point $(-1/3, 11/3)$ is the saddle point because $D(-1/3, 11/3) < 0$.

- **3. (15%) Evaluate the integral by reversing the order of integration**

$$\int_0^4 \int_{\sqrt{x}}^2 \sin y^3 dydx$$

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◦ Sol : The region of integration is shown in the figure.

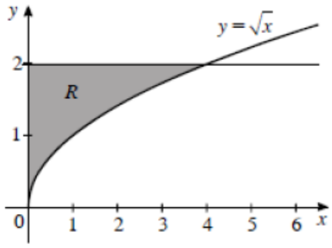

Fig. 1: Figure of integration region

$$\begin{aligned}\Rightarrow \int_0^4 \int_{\sqrt{x}}^2 \sin y^3 \ dydx &= \int_0^2 \int_0^{y^2} \sin y^3 \ dxdy = \int_0^2 \left[x \sin y^3\right]_0^{y^2} dy \\ &= \int_0^2 y^2 \sin y^3 \ dy = -\frac{1}{3}\cos(y^3)\Big|_0^2 = \frac{1-\cos 8}{3}\end{aligned}$$

- **4. (15%) Use the method of change of variables to evaluate**

$$\int\int_R (x+y)dA$$

where R **is the parallelogram with vertices** $(0,0)$ **,**$(2,0)$ **,** $(3,1)$ **and** $(1,1)$ **.**

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Let $x = u+v$ and $y = v$ and this transformation maps region R into region S as shown in the figure.

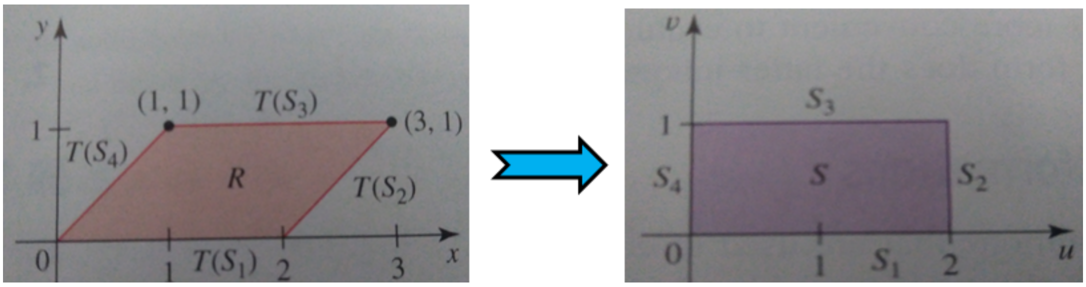

Fig. 2: Figure of integration region

And the Jacobian of the transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

Hence we have

$$\begin{aligned} \int\int_R (x+y)dA &= \int\int_S ((u+v)+v)\cdot 1 \cdot dudv \\ &= \int_0^1 \int_0^2 (u+2v)dudv = \int_0^1 \left[\frac{1}{2}u^2 + 2uv\right]_0^2 dc \\ &= \int_0^1 2+4v\ dv = 2v + 2v^2\Big|_0^1 = 4 \end{aligned}$$

- **5. (10%) Let $\omega = x^2y + y^2z^3$, where $x = r\cos s$, $y = r\sin s$ and $z = re^s$ Use the method of the chain rule to find the value $\partial\omega/\partial s$ when $r = 1$ and $s = 0$**

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- Sol :

$$\begin{aligned}\frac{\partial\omega}{\partial s} = \frac{\partial\omega}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial\omega}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial\omega}{\partial z}\frac{\partial z}{\partial s} &= 2xy(-r\sin s) + (x^2 + 2yz^3)\cdot(r\cos s) + (3y^2z^2)\cdot re^s \\ &= -2rxy\sin s + x^2 r\cos s + 2yz^3 r\cos s + 3y^2z^2re^s\end{aligned}$$

when $r = 1$ and $s = 0$, we have $x = 1$, $y = 0$ and $z = 1$, so

$$\frac{\partial\omega}{\partial s} = 0 + 1 + 0 + 0 = 1$$

- **6. (10%) Find the directional derivative of $f(x, y) = e^x\cos(2y)$ at the point $(0, \pi/4)$ in the direction $\vec{v} = 2\hat{i} + 3\hat{j}$.**

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- Sol :

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\ \hat{i} + \frac{\partial f}{\partial y}\ \hat{j} = e^x\cos(2y)\ \hat{i} - 2e^x\sin(2y)\ \hat{j}$$

And the unit vector of $\vec{v}$ is given by

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{13}}(2\hat{i} + 3\hat{j}) = \frac{2}{\sqrt{13}}\ \hat{i} + \frac{3}{\sqrt{13}}\ \hat{j}$$

Hence the directional derivative of f at the point $(0, \pi/4)$ n the direction $\vec{v}$ is

$$D_{\vec{u}}f(x, y)\Big|_{(0,\pi/4)} = \nabla f(x, y)\cdot\vec{u}\Big|_{(0,\pi/4)} = \frac{2}{\sqrt{13}}\cdot e^x\cos(2y) - \frac{6}{\sqrt{13}}e^x\sin(2y)\Big|_{(0,\pi/4)} = \frac{-6}{\sqrt{13}} = \frac{-6\sqrt{13}}{13}$$

- **7. (10%) Find equations of the tangent plane and normal line to the graph of the function f define by $f(x, y) = 4x^2 + y^2 + 2$ at the point where $x = 1$ and $y = 1$**

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- Sol :

The surface is a paraboloid. The equation can be written as the form

$$z = f(x, y) = 4x^2 + y^2 + 2 \;\Rightarrow\; F(x, y, z) = z - f(x, y) = 0$$

where

$$F(x, y, z) = z - 4x^2 - y^2 - 2$$

Then

$$F_x = \frac{\partial F}{\partial x} = -8x \quad ; \quad F_y = \frac{\partial F}{\partial y} = -2y \quad ; \quad F_z = \frac{\partial F}{\partial z} = 1$$

If $x = 1$, $y = 1$ then $z = f(x, y) = 7$ on the paraboloid.

At point $(1, 1, 7)$ we have

$$F_x = -8 \quad ; \quad F_y = -2 \quad ; \quad F_z = 1$$

Hence we know the normal vector of the tangent plane at $(1, 1, 7)$ is $(-8, -2, 1)$.

Then we have the equation of the plane

$$-8(x - 1) - 2(y - 1) + (z - 7) = 0 \;\Rightarrow\; 8x + 2y - z = 3$$

And the equation of the normal line is given by

$$x = 1 - 8t \quad ; \quad y = 1 - 2t \quad ; \quad z = 1 + t \quad ; \quad t \neq 0 \;,\; t \in \mathbb{R}$$

$$\text{or} \qquad \frac{x - 1}{-8} = \frac{y - 1}{-2} = \frac{z - 7}{1}$$

- **8. (10%) Find the maximum and minimum values of the function** $f(x, y) = x^2 - 2y$ **subject to** $x^2 + y^2 = 9$**.**

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- Sol : Constraint equation is $g(x, y) = x^2 + y^2 = 9$

$$\nabla f = 2x\ \hat{i} - 2\ \hat{j} \quad ; \quad \nabla g = 2x\ \hat{i} + 2y\ \hat{j}$$

The equation $\nabla f = \lambda \nabla g$ becomes

$$2x\ \hat{i} - 2\ \hat{j} = 2x\lambda\ \hat{i} + 2\lambda y\ \hat{j}$$

so we have

$$2x = 2x\lambda \quad - - - - - \ (1)$$

$$-2 = 2\lambda y$$

$$x^2 + y^2 = 9$$

From (1) we have $x = 0$ or $\lambda = 1$.

If $x = 0$ then $y = \pm 3$, so we have $f(x, y) = -6$ or $f(x, y) = 6$.

If $\lambda = 1$ then $y = -1$, $x = \pm 2\sqrt{2}$, so we have $f(x, y) = 10$.

Hence we know that

the maximum value of f is 10

the minimum value of f is -6