聯合微積分 作業解答 4.3 4.4

4.3 Area

Use the rules of summation and the summation formulas to evaluate the sum.

34. $\sum_{k=1}^{40} k(k^2-k)$

$$\begin{aligned}\sum_{k=1}^{40} k(k^2-k) &= \sum_{k=1}^{40} k^3 - k^2 \\ &= \sum_{k=1}^{40} k^3 - \sum_{k=1}^{40} k^2 \\ &= [\frac{40\cdot(40+1)}{2}]^2 - \frac{40\cdot(40+1)(2\cdot 40+1)}{6} \\ &= 650260\end{aligned}$$

36. $\sum_{k=1}^{n} \frac{1}{n^2}(2k+1)$

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{n^2}(2k+1) &= \frac{1}{n^2}\sum_{k=1}^{n}(2k+1) \\ &= \frac{1}{n^2}[2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1] \\ &= \frac{1}{n^2}[2\frac{n(n+1)}{2} + n] \\ &= \frac{n+2}{n}\end{aligned}$$

Evaluate the limit after first finding the sum using the summation formulas.

39. $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2k}{n^2}$

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2k}{n^2} &= \lim_{n\to\infty}\frac{2}{n^2}\sum_{k=1}^{n} k \\ &= \lim_{n\to\infty}\frac{2}{n^2}\frac{n(n+1)}{2} \\ &= \lim_{n\to\infty}\frac{n+1}{n} \\ &= 1\end{aligned}$$

57.(a) Show that the area of the polygon is $A_n = \frac{1}{2}nr^2\sin(\frac{2\pi}{n})$

圓內接正 n 邊形的頂點與圓心的連線，將圓 n 等分，

並將圓內接正 n 邊形切成 n 個頂角為 $\frac{2\pi}{n}$ ，兩腰長為 r 的等腰三角形

則圓內接正 n 邊形的面積即為 $\frac{1}{2}r^2\sin(\frac{2\pi}{n})$

57.(b) Evaluate $\lim_{n\to\infty} A_n$ to obtain the area of the circle $A = \pi r^2$.

$$\begin{aligned}\lim_{n\to\infty} A_n &= \lim_{n\to\infty}\frac{1}{2}nr^2\sin(\frac{2\pi}{n})\\ &= r^2\lim_{n\to\infty}\frac{n}{2}\sin(\frac{2\pi}{n})\\ &= r^2\lim_{n\to\infty}\pi\cdot\frac{n}{2\pi}\sin(\frac{2\pi}{n})\\ &= \pi r^2\lim_{n\to\infty}\frac{\sin(\frac{2\pi}{n})}{\frac{2\pi}{n}}\\ &= \pi r^2\lim_{x\to 0}\frac{\sin x}{x}\\ &= \pi r^2\end{aligned}$$

58.(a) Show that the perimeter of the polygon is $C_n = 2nr\sin(\frac{\pi}{n})$.

圓內接正 n 邊形的頂點與圓心的連線，將圓 n 等分，

並將圓內接正 n 邊形切成 n 個頂角為 $\frac{2\pi}{n}$，兩腰長為 r 的等腰三角形

等腰三角形底邊的長度是 $2r\sin(\frac{\pi}{n})$

圓內接正 n 邊形的周長即為 $2nr\sin(\frac{\pi}{n})$

58.(b) Evaluate $\lim_{n\to\infty} C_n$ to obtain the circumference of the circle $C = 2\pi r$.

$$\begin{aligned}\lim_{n\to\infty} C_n &= \lim_{n\to\infty} 2nr\sin(\frac{\pi}{n})\\ &= 2r\lim_{n\to\infty} n\sin(\frac{\pi}{n})\\ &= 2r\lim_{n\to\infty}\pi\cdot\frac{n}{\pi}\sin(\frac{\pi}{n})\\ &= 2\pi r\lim_{n\to\infty}\frac{\sin(\frac{\pi}{n})}{\frac{\pi}{n}}\\ &= 2\pi r\lim_{x\to 0}\frac{\sin x}{x}\\ &= 2\pi r\end{aligned}$$

4.4 The Definite Integral

12. Use p.387 Equation(2) to evaluate the intergral.

$$
\begin{aligned}
\int_{-2}^{1}(x^3+2x)dx &= \lim_{n\to\infty}\sum_{k=1}^{n}\{[(-2+\frac{3k}{n})^3+2(-2+\frac{3k}{n})]\cdot\frac{1-(-2)}{n}\} \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{3}{n}[(\frac{3}{n}k)^3-3\cdot 2(\frac{3}{n}k)^2+3\cdot 2^2(\frac{3}{n}k)-8+2(\frac{3}{n}k)-4] \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}(\frac{3^4}{n^4}k^3-\frac{2\cdot 3^4}{n^3}k^2+\frac{14\cdot 3^2}{n^2}k-12\cdot\frac{3}{n}) \\
&= \lim_{n\to\infty}[\frac{81}{n^4}\cdot\frac{n^2(n+1)^2}{4}-\frac{2\cdot 3^4}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{14\cdot 3^2}{n^2}\cdot\frac{n(n+1)}{2}-\frac{36}{n}\cdot n] \\
&= \frac{81}{4}-54+63-36 \\
&= \frac{-27}{4}
\end{aligned}
$$

The given expression is the limit of a Riemann sum of a function f on $[a,b]$.
Write this expression as a definite integral on $[a,b]$.

14. $\lim_{n\to\infty}\sum_{k=1}^{n}2c_k(1-c_k)^2\Delta x$,$[0,3]$

設 $f(x)=2x(1-x)^2$, 則 $f(c_k)=2c_k(1-c_k)^2$, $\Delta x=\frac{(3-0)}{n}$

$$
\begin{aligned}
\lim_{n\to\infty}\sum_{k=1}^{n}f(c_k)\Delta x &= \lim_{n\to\infty}\sum_{k=1}^{n}2c_k(1-c_k)^2\Delta x \\
&= \int_{0}^{3}2x(1-x)^2dx
\end{aligned}
$$

Ans. $\int_0^3 2x(1-x)^2dx$

16. $\lim_{n\to\infty}\sum_{k=1}^{n}c_k(\cos c_k)\Delta x$,$[0,\frac{\pi}{2}]$

設 $f(x)=x\cos x$, 則 $f(c_k)=c_k(\cos c_k)$,$\Delta x=\frac{\frac{\pi}{2}-0}{n}$

$$
\begin{aligned}
\lim_{n\to\infty}\sum_{k=1}^{n}f(c_k)\Delta x &= \lim_{n\to\infty}\sum_{k=1}^{n}c_k(\cos c_k)\Delta x \\
&= \int_{0}^{\frac{\pi}{2}}x\cos x dx
\end{aligned}
$$

Ans. $\int_0^{\frac{\pi}{2}} x\cos x dx$

48. Suppose that f is continuous on $[a, b]$ .Prove that

$$|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$$

$$\because \ -|f(x)| \leq f(x) \leq |f(x)| \ on \ [a, b]$$
$$\therefore \ -\int_a^b |f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx$$
$$\Rightarrow \ |\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$$

49. Use the result of Exercise 48 to show that

$$|\int_a^b x \sin 2x dx| \leq \frac{1}{2}(b^2 - a^2)$$

where $0 \leq a < b$.

$$\begin{aligned} \int_a^b x \sin 2x dx| &\leq \int_a^b |x \sin 2x| dx \\ &\leq \int_a^b x dx \\ &= \frac{1}{2}(b^2 - a^2) \end{aligned}$$