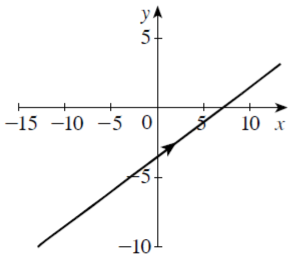
10.2 Plane Curves and Parametric Equations

1. a. $\left.\begin{array}{l} x = 2t + 1 \\ y + 3 = t \end{array}\right\} \Rightarrow x = 2(y+3) + 1$ or

$x - 2y - 7 = 0$

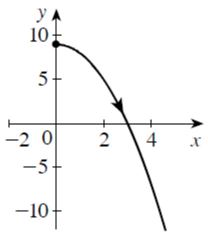
b.



The orientation is found by observing that as t increases, so does x.

3. a. $\left.\begin{array}{l} x = \sqrt{t} \\ y = 9 - t \end{array}\right\} \Rightarrow y = 9 - x^2, x \geq 0$

b.



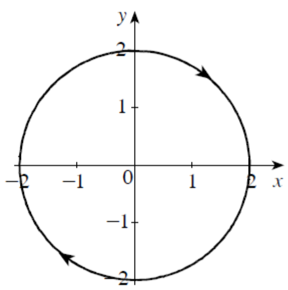
The orientation is found by observing that as t increases, so does x.

9. a. $\left.\begin{array}{l} x = 2\sin\theta \\ y = 2\cos\theta \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \sin\theta = \frac{1}{2}x \\ \cos\theta = \frac{1}{2}y \end{array}\right\} \Rightarrow$

$\left(\frac{1}{2}x\right)^2 + \left(\frac{1}{2}y\right)^2 = \sin^2\theta + \cos^2\theta = 1$, so

$x^2 + y^2 = 4$.

b.



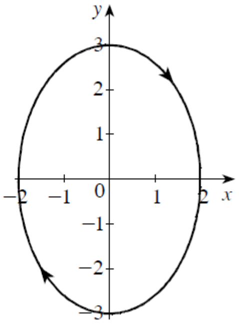
Observe that as θ increases from 0 to 2π, the curve C is traced once in a clockwise direction starting from the point (0, 2).

11. a. $\left.\begin{array}{l} x = 2\sin\theta \\ y = 3\cos\theta \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \sin\theta = \frac{1}{2}x \\ \cos\theta = \frac{1}{3}y \end{array}\right\} \Rightarrow$

$\left(\frac{1}{2}x\right)^2 + \left(\frac{1}{3}y\right)^2 = \sin^2\theta + \cos^2\theta = 1 \Rightarrow$

$\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$

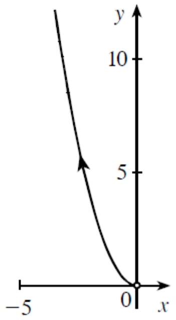
b.



21. a. $\left.\begin{array}{l} x = -e^t \\ y = e^{2t} \end{array}\right\} \Rightarrow y = \left(e^t\right)^2 = (-x)^2 \Rightarrow y = x^2,$

$x < 0$

b.



10.3 The Calculus of Parametric Equations

4. $x = e^{2t}$, $y = \ln t \Rightarrow \frac{dx}{dt} = 2e^{2t}$ and $\frac{dy}{dt} = 1/t$. The slope of the tangent line at $t = 1$ is $\left.\frac{dy}{dx}\right|_{t=1} = \left.\frac{dy/dt}{dx/dt}\right|_{t=1} = \left.\frac{1/t}{2e^{2t}}\right|_{t=1} = \frac{1}{2e^2}$.

6. $x = 2(\theta - \sin\theta)$, $y = 2(1 - \cos\theta) \Rightarrow \frac{dx}{d\theta} = 2(1 - \cos\theta)$ and $\frac{dy}{d\theta} = 2\sin\theta$. The slope of the tangent line at $\theta = \frac{\pi}{6}$ is $\left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\pi/6} = \left.\frac{2\sin\theta}{2(1-\cos\theta)}\right|_{\theta=\pi/6} = \frac{1}{2\left(1 - \frac{\sqrt{3}}{2}\right)} = \frac{1}{2-\sqrt{3}} = 2 + \sqrt{3}.$

8. $x = \theta\cos\theta$, $y = \theta\sin\theta \Rightarrow \frac{dx}{d\theta} = \cos\theta - \theta\sin\theta$ and $\frac{dy}{d\theta} = \sin\theta + \theta\cos\theta$. The point of tangency is $\left(x\left(\frac{\pi}{2}\right), y\left(\frac{\pi}{2}\right)\right) = \left(0, \frac{\pi}{2}\right)$ and the slope of the tangent line is $\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\pi/2} = \left.\frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}\right|_{\theta=\pi/2} = -\frac{2}{\pi}$, so an equation is $y - \frac{\pi}{2} = -\frac{2}{\pi}(x - 0)$ or $y = -\frac{2}{\pi}x + \frac{\pi}{2}$.

20. $x = \sin 2t,\ y = \cos 2t \Rightarrow \dfrac{dx}{dt} = 2\cos 2t$ and $\dfrac{dy}{dt} = -2\sin 2t$, so $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-2\sin 2t}{2\cos 2t} = -\tan 2t$ and

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(-\tan 2t)}{2\cos 2t} = \frac{-2\sec^2 2t}{2\cos 2t} = -\sec^3 2t.$$

22. $x = e^{-t},\ y = e^{2t} \Rightarrow \dfrac{dx}{dt} = -e^{-t}$ and $\dfrac{dy}{dt} = 2e^{2t}$, so $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2e^{2t}}{-e^{-t}} = -2e^{3t}$ and

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(-2e^{3t}\right)}{-e^{-t}} = \frac{-6e^{3t}}{-e^{-t}} = 6e^{4t}.$$

24. $x = \sqrt{t^2+1},\ y = t\ln t \Rightarrow \dfrac{dx}{dt} = \dfrac{t}{\sqrt{t^2+1}}$ and $\dfrac{dy}{dt} = \ln t + 1$, so $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\ln t + 1}{t\Big/\sqrt{t^2+1}} = \dfrac{\sqrt{t^2+1}\,(\ln t+1)}{t}$. Next,

$$\frac{d}{dt}\left[\frac{\left(t^2+1\right)^{1/2}(\ln t+1)}{t}\right] = \frac{t\left[\frac{1}{2}\left(t^2+1\right)^{-1/2}(2t)(\ln t+1) + \frac{\left(t^2+1\right)^{1/2}}{t}\right] - \left(t^2+1\right)^{1/2}(\ln t+1)}{t^2} = \frac{t^2 - \ln t}{t^2\sqrt{t^2+1}},$$

so $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{t^2-\ln t}{t^2\sqrt{t^2+1}}\cdot\frac{\sqrt{t^2+1}}{t} = \frac{t^2-\ln t}{t^3}.$$

28. $x = \displaystyle\int_1^t \frac{\sin u}{u}\,du,\ y = \int_2^{\ln t} e^u\,du$. Using the Fundamental Theorem of Calculus, Part I, we have

$\dfrac{dx}{dt} = \dfrac{d}{dt}\displaystyle\int_1^t \frac{\sin u}{u}\,du = \frac{\sin t}{t}$ and $\dfrac{dy}{dt} = \dfrac{d}{dt}\displaystyle\int_2^{\ln t} e^u\,du = e^{\ln t}\cdot\frac{d}{dt}\ln t = \frac{t}{t} = 1$, so $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1}{\frac{\sin t}{t}} = \dfrac{t}{\sin t}$ and

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{1}{\frac{\sin t}{t}}\cdot\frac{d}{dt}\left(\frac{t}{\sin t}\right) = \frac{t}{\sin t}\cdot\frac{\sin t - t\cos t}{(\sin t)^2} = \frac{t\,(\sin t - t\cos t)}{\sin^3 t}.$$