13. 6

3. Here $\mathbf{u} = \cos\frac{\pi}{2}\,\mathbf{i} + \sin\frac{\pi}{2}\,\mathbf{j} = \mathbf{j}$, so $D_{\mathbf{u}} f(3,0) = \dfrac{\partial f}{\partial y}(3,0) = (x+1)\,e^{y}\big|_{(3,0)} = 4$.

7. $f_x(x,y) = \frac{\partial}{\partial x}(x\sin y + y\cos x) = \sin y - y\sin x$ and $f_y(x,y) = x\cos y + \cos x$, so

$\nabla f\left(\frac{\pi}{4},\frac{\pi}{2}\right) = \left[(\sin y - y\sin x)\,\mathbf{i} + (x\cos y + \cos x)\,\mathbf{j}\right]_{(\pi/4,\pi/2)} = \left[1 - \frac{\pi}{2}\left(\frac{\sqrt{2}}{2}\right)\right]\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} = \frac{4-\sqrt{2}\pi}{4}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$.

15. Here $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{-\mathbf{i}+3\mathbf{j}}{\sqrt{(-1)^2+3^2}} = -\frac{\sqrt{10}}{10}\mathbf{i} + \frac{3\sqrt{10}}{10}\mathbf{j}$, $f_x(x,y) = -\dfrac{2y}{(x-y)^2}$, and $f_y(x,y) = \dfrac{2x}{(x-y)^2}$, so

$D_{\mathbf{u}} f(2,1) = f_x(2,1)\left(-\frac{\sqrt{10}}{10}\right) + f_y(2,1)\left(\frac{3\sqrt{10}}{10}\right) = -2\left(-\frac{\sqrt{10}}{10}\right) + 4\left(\frac{3\sqrt{10}}{10}\right) = \frac{7\sqrt{10}}{5}$.

30. Here $\mathbf{u} = \dfrac{\overrightarrow{PQ}}{\left|\overrightarrow{PQ}\right|} = \dfrac{-3\mathbf{i}+2\mathbf{j}}{\sqrt{(-3)^2+2^2}} = -\frac{3\sqrt{13}}{13}\mathbf{i} + \frac{2\sqrt{13}}{13}\mathbf{j}$, $f_x(x,y) = e^{-y}$, and $f_y(x,y) = -xe^{-y}$, so

$D_{\mathbf{u}} f(2,0) = f_x(2,0)\left(-\frac{3\sqrt{13}}{13}\right) + f_y(2,0)\left(\frac{2\sqrt{13}}{13}\right) = 1\left(-\frac{3\sqrt{13}}{13}\right) - 2\left(\frac{2\sqrt{13}}{13}\right) = -\frac{7\sqrt{13}}{13}$.

39. $\nabla f(x,y,z) = \dfrac{1}{y}\mathbf{i} + \left(-\dfrac{x}{y^2} + \dfrac{1}{z}\right)\mathbf{j} - \dfrac{y}{z^2}\mathbf{k} \Rightarrow \nabla f(1,-1,2) = -\mathbf{i} - \frac{1}{2}\mathbf{j} + \frac{1}{4}\mathbf{k}$ and a vector in the desired direction is $\mathbf{v} = -4\nabla f(1,-1,2) = 4\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. The maximum rate of decrease of f at P is $|-\nabla f(1,-1,2)| = \sqrt{(-1)^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2} = \frac{\sqrt{21}}{4}$.

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5. $F(x, y) = \frac{x^2}{9} + \frac{y^2}{16} \Rightarrow \nabla F\left(\frac{3\sqrt{3}}{2}, 2\right) = \left(\frac{2}{9}x\mathbf{i} + \frac{1}{8}y\mathbf{j}\right)\Big|_{(3\sqrt{3}/2,2)} = \frac{\sqrt{3}}{3}\mathbf{i} + \frac{1}{4}\mathbf{j}$ is normal to the level curve $F(x, y) = \frac{x^2}{9} + \frac{y^2}{16} = 1$ at $\left(\frac{3\sqrt{3}}{2}, 2\right)$. From this, we see that the slope of the required normal line is $m = \frac{3}{4\sqrt{3}} = \frac{\sqrt{3}}{4}$. So an equation of the required normal line is $y - 2 = \frac{\sqrt{3}}{4}\left(x - \frac{3\sqrt{3}}{2}\right) \Leftrightarrow y = \frac{\sqrt{3}}{4}x + \frac{7}{8}$. The slope of the required tangent line is $m = -\frac{4}{\sqrt{3}} = -\frac{4\sqrt{3}}{3}$, and so an equation of the tangent line is $y - 2 = -\frac{4\sqrt{3}}{3}\left(x - \frac{3\sqrt{3}}{2}\right) \Leftrightarrow y = -\frac{4\sqrt{3}}{3}x + 8$.

27. $F(x, y, z) = \ln(xy + 1) - z = 0 \Rightarrow \nabla F(3, 0, 0) = \left(\frac{y}{xy+1}\mathbf{i} + \frac{x}{xy+1}\mathbf{j} - \mathbf{k}\right)\Big|_{(3,0,0)} = 3\mathbf{j} - \mathbf{k}$, so an equation of the tangent plane at $(3, 0, 0)$ is $3(y - 0) - (z - 0) = 0 \Leftrightarrow 3y - z = 0$. Equations of the normal line passing through $(3, 0, 0)$ are $x = 3$, $\frac{y}{3} = \frac{z}{-1}$.

31. $F(x, y, z) = \sin xy + 3z - 3 = 0 \Rightarrow \nabla F(0, 3, 1) = (y\cos xy\,\mathbf{i} + x\cos xy\,\mathbf{j} + 3\mathbf{k})|_{(0,3,1)} = 3\mathbf{i} + 3\mathbf{k} = 3(\mathbf{i} + \mathbf{k})$, so an equation of the tangent plane at $(0, 3, 1)$ is $(x - 0) + (z - 1) = 0 \Leftrightarrow x + z = 1$. Equations of the normal line passing through $(0, 3, 1)$ are $\frac{x}{1} = \frac{z-1}{1}$, $y = 3$.

34. $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 \Rightarrow \nabla F(x_0, y_0, z_0) = \frac{2x_0}{a^2}\mathbf{i} + \frac{2y_0}{b^2}\mathbf{j} - \frac{2z_0}{c^2}\mathbf{k}$, so an equation of the tangent plane at (x_0, y_0, z_0) is $\frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) - \frac{2z_0}{c^2}(z - z_0) = 0 \Leftrightarrow \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} - \frac{z_0 z}{c^2} - \left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - \frac{z_0^2}{c^2}\right) = 0$. But (x_0, y_0, z_0) lies on the hyperboloid, so the expression in parentheses is equal to 1 and we have $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} - \frac{zz_0}{c^2} = 1$, as was to be shown.

42. We write $F(x, y, z) = x^2 + y^2 + z^2 - 17$ and $G(x, y, z) = 2x^2 - y + 2z^2 + 2$. The vector in the direction of the normal line to the sphere $F(x, y, z) = 0$ at $(1, 4, 0)$ is $\mathbf{v} = \nabla F(1, 4, 0) = (2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})|_{(1,4,0)} = 2\mathbf{i} + 8\mathbf{j}$ and the vector in the direction of the normal line to the elliptic paraboloid $G(x, y, z) = 0$ at $(1, 4, 0)$ is $\mathbf{w} = \nabla G(1, 4, 0) = (4x\mathbf{i} - \mathbf{j} + 4z\mathbf{k})|_{(1,4,0)} = 4\mathbf{i} - \mathbf{j}$. Since $\mathbf{v} \cdot \mathbf{w} = (2\mathbf{i} + 8\mathbf{j}) \cdot (4\mathbf{i} - \mathbf{j}) = 0$, we see that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, completing the proof.