6.2 Inverse Functions

40. a. $f(g(x)) = f\left(\frac{1}{2}+\sqrt{\frac{5}{4}-x}\right) = -\left(\frac{1}{2}+\sqrt{\frac{5}{4}-x}\right)^2 + \left(\frac{1}{2}+\sqrt{\frac{5}{4}-x}\right) + 1$

$= -\frac{1}{4} - \sqrt{\frac{5}{4}-x} - \frac{5}{4} + x + \frac{1}{2} + \sqrt{\frac{5}{4}-x} + 1 = x$

$g(f(x)) = g\left(-x^2+x+1\right) = \frac{1}{2} + \sqrt{\frac{5}{4}+x^2-x-1} = \frac{1}{2} + \sqrt{x^2-x+\frac{1}{4}}$

$= \frac{1}{2} + \sqrt{\left(x-\frac{1}{2}\right)^2} = \frac{1}{2} + \left(x-\frac{1}{2}\right) = x$

b. $-x^2+x+1 = \frac{1}{2}+\sqrt{\frac{5}{4}-x} \Rightarrow f(x) = g(x)$. The equation is found by solving $y = f(x)$ and $y = g(x)$ simultaneously, but f and g are inverse functions, so their graphs intersect at $y = x$. Setting $f(x) = x$ gives $-x^2+x+1 = x \Rightarrow -x^2+1 = 0 \Rightarrow x = \pm 1$. We reject the negative root since y must be positive, so $x = 1$.

45. a. $f(x) = x^2$ on $[0,\infty) \Rightarrow g(x) = f^{-1}(x) = \sqrt{x}$. Using Formula 3, $g'(x) = \dfrac{1}{f'(g(x))} = \dfrac{1}{2y}\Big|_{y=\sqrt{x}} = \dfrac{1}{2\sqrt{x}}$.

b. From the result of part a, $g(x) = \sqrt{x}$, so $g'(x) = \dfrac{1}{2\sqrt{x}}$.

46. a. $f(x) = x^{1/3} \Rightarrow g(x) = f^{-1}(x) = x^3$. Using Formula 3, $g'(x) = \dfrac{1}{f'(g(x))} = \dfrac{1}{\frac{1}{3}y^{-2/3}}\Bigg|_{y=x^3} = 3\left(x^3\right)^{2/3} = 3x^2$.

b. Since $g(x) = x^3$, we find $g'(x) = 3x^2$.

56. $H'(x) = \dfrac{d}{dx}\left[g(g(x))\right] = g'(g(x))\,g'(x)$, so

$$H'(3) = g'(g(3))\,g'(3) = g'(4)\,g'(3) = \frac{1}{f'(g(4))}\cdot\frac{1}{f'(g(3))} = \frac{1}{f'(5)}\cdot\frac{1}{f'(4)} = \frac{1}{2}\cdot\frac{1}{\frac{1}{2}} = 1.$$

6.3 Exponential Functions

24. $f'(x) = \dfrac{d}{dx}\left(x^2e^{-2x}\right) = x^2\dfrac{d}{dx}\left(e^{-2x}\right) + e^{-2x}\dfrac{d}{dx}\left(x^2\right) = -2x^2e^{-2x} + 2xe^{-2x} = 2xe^{-2x}(1-x)$

35. $y' = \dfrac{d}{dx}\left(e^x+\ln x^2\right)^3 = 3\left(e^x+\ln x^2\right)^2\left(e^x+\dfrac{2}{x}\right) = \dfrac{3}{x}\left(xe^x+2\right)\left(e^x+\ln x^2\right)^2$

41. $xe^{2y} - x^3 + 2y = 5 \Rightarrow e^{2y} + xe^{2y}\left(2y'\right) - 3x^2 + 2y' = 0 \Leftrightarrow y' = \dfrac{3x^2-e^{2y}}{2\left(xe^{2y}+1\right)}$

50. $xe^y + 2x + y = 3 \Rightarrow e^y + xe^y y' + 2 + y' = 0$. Substituting $x = 1$ and $y = 0$ into this equation gives $1 + y' + 2 + y' = 0$ or $y'|_{(1,0)} = -\frac{3}{2}$, so the slope of the required tangent line is $m = -\frac{3}{2}$, and an equation is $y - 0 = -\frac{3}{2}(x-1)$ or $y = -\frac{3}{2}x + \frac{3}{2}$.