聯合微積分 作業解答 3.4 3.5

3.4 Concavity and Inflection Point

Determine where the graph of the function is concave upward and where it is concave downward. Also , find all inflection points of the function.

12. $g(x) = x^3 - 6x^2 + 2x + 3$

$$\begin{aligned} g'(x) &= 3x^2 - 12x + 2 \\ g''(x) &= 6x - 12 \\ &= 6(x-2) \\ g''(x) &> 0, x \in (2, \infty) \\ g''(x) &= 0, x = 2 \\ g''(x) &< 0, x \in (-\infty, 2) \end{aligned}$$

Concave upward : $(2, \infty)$
Concave downward : $(-\infty, 2)$
Inflection points : $x = 2$

20. $g(x) = x + \frac{1}{x}$

$$\begin{aligned} g'(x) &= 1 - \frac{1}{x^2} \\ g''(x) &= 2x^{-3} \\ g''(x) &> 0, x \in (0, \infty) \\ g''(x) &< 0, x \in (-\infty, 0) \end{aligned}$$

Concave upward : $(0, \infty)$
Concave downward : $(-\infty, 0)$
Inflection points : not exist.

Find the relative extrema , if any , of the function. Use the Second Derivative Test , if applicable.

40. $f(x) = 2x^4 - 8x + 4$

$$\begin{aligned} f'(x) &= 8x^3 - 8 \\ &= 8(x-1)(x^2+x+1) \\ f''(x) &= 24x^2 \\ f'(1) &= 0 \\ f''(1) &= 24 > 0 \end{aligned}$$

Relative minimum : $f(1) = -2$

42. $h(t) = t^2 + \frac{1}{t}$

$$\begin{aligned} h'(t) &= 2t - \frac{1}{t^2} \\ h''(t) &= 2 + 2x^{-3} \\ h'(2^{-\frac{1}{3}}) &= 0 \\ h''(2^{-\frac{1}{3}}) &= 6 > 0 \end{aligned}$$

Relative minimum :$h(2^{-\frac{1}{3}}) = 3 \cdot 2^{-\frac{2}{3}}$

47. $f(x) = 2\sin x + \sin 2x, 0 < x < \pi$

$$\begin{aligned} f'(x) &= 2\cos x + 2\cos 2x \\ f''(x) &= -2\sin x - 4\sin 2x \\ f'(\frac{\pi}{3}) &= 2(\cos\frac{\pi}{3} + \cos\frac{2\pi}{3}) \\ &= 2(\frac{1}{2} - \frac{1}{2}) = 0 \\ f''(\frac{\pi}{3}) &= -2\sin\frac{\pi}{3} - 4\sin\frac{2\pi}{3} \\ &= -\sqrt{3} - 2\sqrt{3} = -3\sqrt{3} \ < 0 \end{aligned}$$

Relative maxmum : $f(\frac{\pi}{3}) = \frac{3\sqrt{3}}{2}$

3.5 Limits Involving Infinity ; Asymptotes

Find the limit.

32. $\lim_{t\to-\infty}\frac{2t^2}{\sqrt{t^4+t^2}}$

$$\begin{aligned}\lim_{t\to-\infty}\frac{2t^2}{\sqrt{t^4+t^2}} &= \lim_{t\to-\infty}\frac{2}{\frac{1}{t^2}\cdot\sqrt{t^4+t^2}}\\ &= \lim_{t\to-\infty}\frac{2}{\sqrt{\frac{t^4+t^2}{t^4}}}\\ &= \lim_{t\to-\infty}\frac{2}{\sqrt{1+\frac{1}{t^2}}}\\ &= 2\end{aligned}$$

36. $\lim_{x\to\infty}\frac{x}{3x+\cos x}$

$$\begin{aligned}\lim_{x\to\infty}\frac{x}{3x+\cos x} &= \lim_{x\to\infty}\frac{1}{\frac{1}{x}(3x+\cos x)}\\ &= \lim_{x\to\infty}\frac{1}{3+\frac{\cos x}{x}}\\ &= \frac{1}{3}\end{aligned}$$

Find the horizontal and vertical asymptotes of the graph of the function .

Definition vertical asymptote 鉛直漸進線

The line $x = a$ is a **vertical asymptote** of the graph of a function f if at least one of the following statements is true:

$$\lim_{x\to a^-} f(x) = \infty(or-\infty);\ \lim_{x\to a^+} f(x) = \infty(or-\infty);\ \lim_{x\to a} f(x) = \infty(or-\infty)$$

Definition horizontal asymptote 水平漸進線

The line $y = L$ is a **horizontal asymptote** of the graph of a function f if

$$\lim_{x\to\infty} f(x) = L or \lim_{x\to-\infty} f(x) = L$$

53. $f(x) = \frac{2x}{x^2-x-6}$

$$\frac{2x}{x^2-x-6} = \frac{2x}{(x-3)(x+2)}$$

$$\lim_{x\to\infty}\frac{2x}{x^2-x-6} = 0$$

$$\lim_{x\to-\infty}\frac{2x}{x^2-x-6} = 0$$

$$\lim_{x\to3^+}\frac{2x}{(x-3)(x+2)} = \infty\ or \lim_{x\to3^-}\frac{2x}{(x-3)(x+2)} = -\infty$$

$$\lim_{x\to-2^-}\frac{2x}{(x-3)(x+2)} = \infty\ or \lim_{x\to-2^+}\frac{2x}{(x-3)(x+2)} = -\infty$$

horizontal asymptotes : $y = 0$
vertical asymptotes :$x = -2$ and $x = 3$

54. $h(x) = \frac{2-x^2}{x^2+x}$

$$\lim_{x\to\infty} \frac{2-x^2}{x^2+x} = -1$$
$$\lim_{x\to-\infty} \frac{2-x^2}{x^2+x} = -1$$

$$\lim_{x\to-1^-} \frac{2-x^2}{x(x+1)} = \infty \; or \lim_{x\to-1^+} \frac{2-x^2}{x(x+1)} = -\infty$$
$$\lim_{x\to0^+} \frac{2-x^2}{x(x+1)} = \infty \; or \lim_{x\to0^-} \frac{2-x^2}{x(x+1)} = -\infty$$

horizontal asymptotes : $y = -1$
vertical asymptotes :$x = -1$ and $x = 0$

72. Determine the constants a and b such that

$$\lim_{x\to\infty} (\frac{2x^2+3}{x+1} - ax - b) = 0$$

$$\begin{aligned}
\lim_{x\to\infty} (\frac{2x^2+3}{x+1} - ax - b) &= \lim_{x\to\infty} \frac{2x^2+3-ax^2-ax-bx-b}{x+1} \\
&= \lim_{x\to\infty} \frac{(2-a)x^2-(a+b)x+3-b}{x+1} \\
&= \lim_{x\to\infty} \frac{\frac{1}{x}(2-a)x^2-(a+b)x+3-b]}{\frac{1}{x}(x+1)} \\
&= \lim_{x\to\infty} \frac{(2-a)x-(a+b)+\frac{3-b}{x}}{1\frac{1}{x}} \\
&= 0 \\
&\Rightarrow 2-a=0, a+b=0 \\
&\Rightarrow a=2, b=-2
\end{aligned}$$