聯合微積分 作業解答 2.7 3.1

2.7 Implicit Differentiation

Find dy/dx by implicit differentiation.

6. $x^3y^2 - 2x^2y + 2x = 3$

$$\begin{aligned}
&\frac{d}{dx}(x^3y^2 - 2x^2y + 2x) = \frac{d}{dx}(3)\\
\Rightarrow &\frac{d}{dx}(x^3y^2) - \frac{d}{dx}(2x^2y) + \frac{d}{dx}(2x) = \frac{d}{dx}(3)\\
\Rightarrow &(3x^2y^2 + 2x^3y\frac{dy}{dx}) - (4xy + 2x^2\frac{dy}{dx}) + 2 = 0\\
\Rightarrow &(2x^3y - 2x^2)\frac{dy}{dx} = 4xy - 3x^2y^2 - 2\\
\Rightarrow &\frac{dy}{dx} = \frac{4xy - 3x^2y^2 - 2}{2x^3y - 2x^2}
\end{aligned}$$

Use implicit differentiation to find an equation of the tangent line to the curve at the indicated point.

24. $y = sin(xy)$, $(\frac{\pi}{2}, 1)$

$$\begin{aligned}
&\frac{dy}{dx} = \frac{d}{dx}(\sin xy)\\
\Rightarrow\ &\frac{dy}{dx} = \cos xy(y + x\frac{dy}{dx})\\
\Rightarrow\ &\frac{dy}{dx} = y\cos xy + x\cos xy\frac{dy}{dx}\\
\Rightarrow\ &\frac{dy}{dx}(1 - x\cos xy) = y\cos xy\\
\Rightarrow\ &\frac{dy}{dx} = \frac{y\cos xy}{1 - x\cos xy}
\end{aligned}$$

$(\frac{\pi}{2}, 1)$ 代入 $\frac{y\cos xy}{1-x\cos xy}$ ，得到 $\frac{1\cdot\cos\frac{\pi}{2}}{1-\frac{\pi}{2}} = 0$

ans: $y = 1$

Find $\frac{d^2y}{dx^2}$ in terms of x and y.

31 $\sin x + \cos y = 1$

$$\frac{d}{dx}(\sin x + \cos y) = \frac{d}{dx}(1)$$
$$\Rightarrow \cos x - \sin y \frac{dy}{dx} = 0$$
$$\Rightarrow \sin y \frac{dy}{dx} = \cos x$$
$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{\sin y}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{\cos x}{\sin y})$$
$$= \frac{-\sin x \sin y - \cos x \cos y \frac{dy}{dx}}{\sin^2 y}$$
$$= \frac{-\sin x}{\sin y} - \frac{\cos^2 x \cos y}{\sin^3 y}$$

32 $\tan y - xy = 0$

$$\frac{d}{dx}(\tan y - xy) = 0$$
$$\Rightarrow \sec^2 y \frac{dy}{dx} - y - x\frac{dy}{dx} = 0$$
$$\Rightarrow (\sec^2 y - x)\frac{dy}{dx} = y$$
$$\Rightarrow \frac{dy}{dx} = \frac{y}{\sec^2 y - x}$$

$$\frac{d}{dx}(\sec^2 y \frac{dy}{dx} - y - x\frac{dy}{dx}) = 0$$
$$\Rightarrow (2\sec^2 y \tan y)(\frac{dy}{dx})^2 + (\sec^2 y)\frac{d^2y}{dx^2} - \frac{dy}{dx} - \frac{dy}{dx} - x\frac{d^2y}{dx^2} = 0$$
$$\Rightarrow (\sec^2 y - x)\frac{d^2y}{dx^2} = 2\frac{dy}{dx} - (2\sec^2 y \tan y)(\frac{dy}{dx})^2$$
$$\Rightarrow (\sec^2 y - x)\frac{d^2y}{dx^2} = 2\frac{y}{\sec^2 y - x} - (2\sec^2 y \tan y)(\frac{y}{\sec^2 y - x})^2$$
$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2y}{(\sec^2 y - x)^2} - \frac{2y^2 \sec^2 y \tan y}{(\sec^2 y - x)^3}$$
$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2y(\sec^2 y - x - y\sec^2 y \tan y)}{(\sec^2 y - x)^3}$$

Find an equation of the tangent line to the given curve at the indicated point.

36. $2y^2 - x^3 - x^2 = 0$; (1,1)

$$\begin{aligned}
&(2y^2 - x^3 - x^2)' = 0 \\
\Rightarrow\ &4yy' - 3x^2 - 2x = 0 \\
\Rightarrow\ &4yy' = 3x^2 + 2x \\
\Rightarrow\ &y' = \frac{3x^2 + 2x}{4y} \\
\Rightarrow\ &y'(1,1) = \frac{3+2}{4} = \frac{5}{4}
\end{aligned}$$

ans : $y - 1 = \frac{5}{4}(x-1)$ or $y = \frac{5}{4}x - \frac{1}{4}$

3.1 Extrema of Function

Find the critical number(s) , if any ,of the function.

30. $g(t) = 2t^3 + 3t^2 - 12t + 4$

$$\begin{aligned}
g'(t) &= 6t^2 + 6t - 12 \\
&= 6(t^2 + t - 2) \\
&= 6(t+2)(t-1)
\end{aligned}$$

$g'(-2) = 0$ and $g'(1) = 0$

Domain: $(-\infty, \infty)$

Ans: $-2, 1$

32. $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x - 3$

$$\begin{aligned}
f'(x) &= x^2 + x + 2 \\
&= x^2 + x + \frac{1}{4} + \frac{3}{4} \\
&= (x + \frac{1}{2})^2 + \frac{3}{4} > 0
\end{aligned}$$

Domain: $(-\infty, \infty)$

Ans: no critical number(s).

Find the absolute maximum and absolute minimum values , if any , of the function.

44. $f(x) = -x^2 + 4x + 3$ on $[-1, 3]$

$$\begin{aligned}
f'(x) &= -2x + 4 \\
&= -2(x-2) \\
\Rightarrow\ &f'(2) = 0
\end{aligned}$$

$f(-1) = -2, f(3) = 6, f(2) = 7$

max: 7
min: -2

50. $g(u) = \frac{\sqrt{u}}{u^2+1}$ on $[0, 2]$

$$\begin{aligned} g'(x) &= \frac{\frac{1}{2}\frac{1}{\sqrt{u}}(u^2+1) - \sqrt{u}(2u)}{(u^2+1)^2} \\ &= \frac{u^2 + 1 - 4u^2}{2\sqrt{u}(u^2+1)^2} \\ &= \frac{1 - 3u^2}{2\sqrt{u}(u^2+1)^2} \end{aligned}$$

$g'(\frac{1}{\sqrt{3}}) = 0, g'(0)$ doesn't exist.

$g(0) = 0, g(2) = \frac{\sqrt{2}}{5}, g(\frac{1}{\sqrt{3}}) = \frac{3^{\frac{3}{4}}}{4}$

max: $\frac{3^{\frac{3}{4}}}{4}$
min: 0

56. $g(x) = x\sqrt{4 - x^2}$ on $[0, 2]$

$$\begin{aligned} g'(x) &= \sqrt{4 - x^2} - \frac{x^2}{\sqrt{4 - x^2}} \\ &= \frac{4 - x^2 - x^2}{\sqrt{4 - x^2}} \\ &= \frac{2(2 - x^2)}{\sqrt{4 - x^2}} \end{aligned}$$

$g'(\sqrt{2}) = 0$;$g'(2), g'(-2)$ doesn't exist.

$g(2) = 0, g(\sqrt{2}) = 2, g(0) = 0$

max: 2
min: 0