6.5 Inverse Trigonometric Functions

33. $f'(x) = \dfrac{d}{dx}\sin^{-1}3x = \dfrac{1}{\sqrt{1-(3x)^2}}\cdot\dfrac{d}{dx}(3x) = \dfrac{3}{\sqrt{1-9x^2}}$

35. $f'(x) = \dfrac{d}{dx}\tan^{-1}x^2 = \dfrac{1}{1+\left(x^2\right)^2}\cdot\dfrac{d}{dx}\left(x^2\right) = \dfrac{2x}{1+x^4}$

37. $g'(t) = \dfrac{d}{dt}\left[t\tan^{-1}3t\right] = \tan^{-1}3t + t\cdot\dfrac{\frac{d}{dt}(3t)}{1+(3t)^2} = \tan^{-1}3t + \dfrac{3t}{1+9t^2}$

39. $f'(u) = \dfrac{d}{du}\sec^{-1}2u = \dfrac{\frac{d}{du}(2u)}{|2u|\sqrt{(2u)^2-1}} = \dfrac{1}{|u|\sqrt{4u^2-1}}$

51. $f'(x) = \dfrac{d}{dx}\sin^{-1}\left(e^{2x}\right) = \dfrac{1}{\sqrt{1-\left(e^{2x}\right)^2}}\dfrac{d}{dx}e^{2x} = \dfrac{2e^{2x}}{\sqrt{1-e^{4x}}}$

65. $\displaystyle\int\frac{dx}{\sqrt{16-x^2}} = \int\frac{dx}{\sqrt{16\left[1-(x/4)^2\right]}} = \frac{1}{4}\int\frac{dx}{\sqrt{1-(x/4)^2}}$. Let $u = \frac{1}{4}x$, so $du = \frac{1}{4}dx$. Then

$$\int\frac{dx}{\sqrt{16-x^2}} = \int\frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C = \sin^{-1}\left(\tfrac{1}{4}x\right) + C.$$

67. $\displaystyle\int_0^{1/2}\frac{dx}{1+4x^2} = \int_0^{1/2}\frac{dx}{1+(2x)^2}$. Let $u = 2x$, so $du = 2\,dx$, $x = 0 \Rightarrow u = 0$, and $x = \frac{1}{2} \Rightarrow u = 1$. Then

$$\int_0^{1/2}\frac{dx}{1+4x^2} = \frac{1}{2}\int_0^1\frac{du}{1+u^2} = \left.\frac{\tan^{-1}u}{2}\right|_0^1 = \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}.$$

69. Let $u = \frac{1}{9}x^2$, so $du = \frac{2}{9}x\,dx$. Then

$$\int\frac{dx}{x\sqrt{x^4-81}} = \frac{1}{9}\int\frac{x\,dx}{x^2\sqrt{\left(x^2/9\right)^2-1}} = \frac{1}{9}\cdot\frac{9}{2}\int\frac{du}{9u\sqrt{u^2-1}} = \frac{1}{18}\sec^{-1}u + C = \frac{1}{18}\sec^{-1}\left(\frac{x^2}{9}\right) + C.$$

71. $\displaystyle\int\frac{dt}{t\sqrt{t^6-16}} = \int\frac{dt}{t\sqrt{16\left[\left(t^6/16\right)-1\right]}} = \frac{1}{4}\int\frac{dt}{t\sqrt{\left(t^3/4\right)^2-1}}$. Let $u = \dfrac{t^3}{4}$, so $du = \dfrac{3t^2\,dt}{4}$. Then

$$\int\frac{dt}{t\sqrt{t^6-16}} = \frac{1}{4}\int\frac{t^2\,dt}{t^3\sqrt{\left(t^3/4\right)^2-1}} = \frac{1}{4}\cdot\frac{4}{3}\cdot\frac{1}{4}\int\frac{du}{u\sqrt{u^2-1}} = \frac{1}{12}\sec^{-1}|u| + C = \frac{1}{12}\sec^{-1}\frac{\left|t^3\right|}{4} + C.$$

73. $\displaystyle\int_0^1 \frac{e^{2x}}{1+e^{4x}}\,dx = \int_0^1 \frac{e^{2x}}{1+\left(e^{2x}\right)^2}\,dx$. Let $u = e^{2x}$, so $du = 2e^{2x}\,dx$, $x = 0 \Rightarrow u = 1$, and $x = 1 \Rightarrow u = e^2$. Then

$$\int_0^1 \frac{e^{2x}}{1+e^{4x}}\,dx = \frac{1}{2}\int_1^{e^2} \frac{du}{1+u^2} = \tfrac{1}{2}\tan^{-1} u\Big|_1^{e^2} = \tfrac{1}{2}\left(\tan^{-1} e^2 - \tfrac{\pi}{4}\right).$$

75. $\displaystyle\int \frac{\sin x}{\sqrt{4-\cos^2 x}}\,dx = \frac{1}{2}\int \frac{\sin x}{\sqrt{1-\left(\frac{\cos x}{2}\right)^2}}\,dx$. Let $u = \frac{1}{2}\cos x$, so $du = -\frac{1}{2}\sin x\,dx$. Then

$$\int \frac{\sin x}{\sqrt{4-\cos^2 x}}\,dx = \frac{1}{2}(-2)\int \frac{du}{\sqrt{1-u^2}} = -\sin^{-1} u + C = -\sin^{-1}\left(\frac{\cos x}{2}\right) + C.$$

77. $\displaystyle\int_0^1 \frac{x^3}{1+x^8}\,dx = \int_0^1 \frac{x^3}{1+\left(x^4\right)^2}\,dx$. Let $u = x^4$, so $du = 4x^3\,dx$, $x = 0 \Rightarrow u = 0$, and $x = 1 \Rightarrow u = 1$. Then

$$\int_0^1 \frac{x^3}{1+x^8}\,dx = \frac{1}{4}\int_0^1 \frac{du}{1+u^2} = \tfrac{1}{4}\tan^{-1} u\Big|_0^1 = \tfrac{1}{4}\left(\tfrac{\pi}{4}\right) = \tfrac{\pi}{16}.$$

79. Let $u = \sin^{-1} x$, so $du = \dfrac{dx}{\sqrt{1-x^2}}$, $x = 0 \Rightarrow u = 0$, and $x = \frac{\sqrt{3}}{2} \Rightarrow u = \frac{\pi}{3}$. Then

$$\int_0^{\sqrt{3}/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/3} u\,du = \frac{u^2}{2}\Big|_0^{\pi/3} = \frac{\pi^2}{18}.$$

81. Let $u = \tan^{-1} x$, so $du = \dfrac{dx}{1+x^2}$. Then $\displaystyle\int \frac{\tan^{-1} x}{1+x^2}\,dx = \int u\,du = \tfrac{1}{2}u^2 + C = \tfrac{1}{2}\left(\tan^{-1} x\right)^2 + C.$

83. Let $u = \sqrt{x}$, so $du = \frac{1}{2\sqrt{x}}\,dx$. Then $\displaystyle\int \frac{dx}{\sqrt{x}\,(4+x)} = \frac{1}{4}\int \frac{dx}{\sqrt{x}\left(1+\frac{1}{4}x\right)} = \frac{1}{4}\cdot 2\int \frac{du}{1+\frac{1}{4}u^2} = \frac{1}{2}\int \frac{du}{1+\left(\frac{1}{2}u\right)^2}$. Next, let $v = \frac{1}{2}u$, so $dv = \frac{1}{2}\,du$. Then $\displaystyle\int \frac{dx}{\sqrt{x}\,(4+x)} = \frac{1}{2}\cdot 2\int \frac{dv}{1+v^2} = \tan^{-1} v + C = \tan^{-1}\left(\tfrac{1}{2}u\right) + C = \tan^{-1}\left(\tfrac{\sqrt{x}}{2}\right) + C.$

85. $\displaystyle\int \frac{dx}{4+(x-2)^2} = \frac{1}{4}\int \frac{dx}{1+\left(\frac{x-2}{2}\right)^2}$. Let $u = \frac{1}{2}(x-2)$, so $du = \frac{1}{2}\,dx$. Then

$$\int \frac{dx}{4+(x-2)^2} = \frac{1}{4}\cdot 2\int \frac{du}{1+u^2} = \tfrac{1}{2}\tan^{-1} u + C = \tfrac{1}{2}\tan^{-1}\left(\tfrac{x-2}{2}\right) + C.$$

6.7 Indeterminate Forms and l'Hôpital's Rule

1. $\displaystyle\lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{1}{2x} = \frac{1}{2}$

3. $\displaystyle\lim_{x\to 2} \frac{x^3-8}{x-2} = \lim_{x\to 2} \frac{3x^2}{1} = 12$

5. $\displaystyle \lim_{x\to 0}\frac{e^x-1}{x^2+x}=\lim_{x\to 0}\frac{e^x}{2x+1}=1$

7. $\displaystyle \lim_{t\to\pi}\frac{\sin t}{\pi-t}=\lim_{t\to\pi}\frac{\cos t}{-1}=1$

11. $\displaystyle \lim_{x\to\infty}\frac{x+\cos x}{2x+1}=\lim_{x\to\infty}\frac{1-\sin x}{2}$. Observe that the limit on the right-hand side does not exist, so l'Hôpital's Rule does not apply. We evaluate the limit as follows: $\displaystyle \lim_{x\to\infty}\frac{x+\cos x}{2x+1}=\lim_{x\to\infty}\frac{1+\frac{\cos x}{x}}{2+\frac{1}{x}}=\frac{1}{2}$.

15. $\displaystyle \lim_{x\to\infty}\frac{\sqrt{x}}{\ln x}=\lim_{x\to\infty}\frac{1/\left(2\sqrt{x}\right)}{1/x}=\lim_{x\to\infty}\frac{\sqrt{x}}{2}=\infty$

19. $\displaystyle \lim_{x\to\infty}\frac{\ln\left(1+e^x\right)}{x^2}=\lim_{x\to\infty}\frac{e^x/\left(1+e^x\right)}{2x}=\lim_{x\to\infty}\frac{1}{(1/e^x+1)(2x)}=0$

37. $\displaystyle \lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x}=\lim_{x\to 1}\frac{1-\frac{1}{x}}{\ln x+\frac{x-1}{x}}=\lim_{x\to 1}\frac{x-1}{x\ln x+x-1}=\lim_{x\to 1}\frac{1}{\ln x+x\left(\frac{1}{x}\right)+1}=\frac{1}{2}$

41. $\displaystyle \lim_{x\to\infty}\left(\frac{1}{x}e^{-x}\right)=\lim_{x\to\infty}\frac{1}{xe^x}=0$

42. $\displaystyle \lim_{x\to 0^+}x^{\sin x}$ is an indeterminate form of type 0^0, so let $y=x^{\sin x}$. Then $\ln y=\ln x^{\sin x}=\sin x\ln x$, so

$$\ln\left(\lim_{x\to 0^+}y\right)=\lim_{x\to 0^+}(\ln y)=\lim_{x\to 0^+}\frac{\ln x}{\csc x}=\lim_{x\to 0^+}\frac{1/x}{-\csc x\cot x}=-\lim_{x\to 0^+}\frac{\sin^2 x}{x\cos x}=-\lim_{x\to 0^+}\left(\frac{\sin x}{x}\tan x\right)$$
$$=1\cdot 0=0$$

Thus, $\displaystyle \lim_{x\to 0^+}y=\lim_{x\to 0^+}x^{\sin x}=e^0=1$.

52. $\displaystyle \lim_{x\to\infty}(1-1/x)^x$ is an indeterminate form of type 1^∞. Let $y=(1-1/x)^x$, so $\ln y=\ln(1-1/x)^x=x\ln(1-1/x)$. Then

$$\ln\left(\lim_{x\to\infty}y\right)=\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\left[x\ln\left(1-\frac{1}{x}\right)\right]=\lim_{x\to\infty}\frac{\ln\left(1-\frac{1}{x}\right)}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\frac{1/x^2}{1-1/x}}{-1/x^2}=\lim_{x\to\infty}\left(\frac{1}{1/x-1}\right)=-1,$$

so $\displaystyle \lim_{x\to\infty}y=\lim_{x\to\infty}(1-1/x)^x=e^{-1}=1/e$.