6.4 General Exponential and Logarithmic Functions

23. $y = x\left(5^{3x}\right) \Rightarrow y' = 1 \cdot 5^{3x} + x \cdot \ln 5\,(3)\left(5^{3x}\right) = [1 + (3\ln 5)\,x]\,5^{3x}$

30. $h(x) = 2^{\tan x} \Rightarrow h'(x) = (\ln 2)\left(2^{\tan x}\right)\sec^2 x$

47. Let $u = 2^x$, so $du = 2^x \ln 2\,dx$. Then $\int 2^x \sin 2^x\,dx = \frac{1}{\ln 2}\int \sin u\,du = -\frac{\cos u}{\ln 2} + C = -\frac{\cos 2^x}{\ln 2} + C$.

48. Let $u = \sqrt{x}$, so $du = \frac{1}{2\sqrt{x}}\,dx$, $x = 1 \Rightarrow u = 1$, and $x = 4 \Rightarrow u = 2$. Thus, $\int_1^4 \frac{3^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int_1^2 3^u\,du = 2 \cdot \left.\frac{3^u}{\ln 3}\right|_1^2 = \frac{12}{\ln 3}$.

6.5 Inverse Trigonometric Functions

47. $g'(t) = \frac{d}{dt}\tan^{-1}\frac{t-1}{t+1} = \frac{\frac{d}{dt}\left(\frac{t-1}{t+1}\right)}{1+\left(\frac{t-1}{t+1}\right)^2} = \frac{\frac{(t+1)-(t-1)}{(t+1)^2}}{1+\frac{(t-1)^2}{(t+1)^2}} = \frac{2}{(t+1)^2+(t-1)^2} = \frac{1}{t^2+1}$

48. $f'(x) = \frac{d}{dx}\cos^{-1}(\sin 2x) = -\frac{\frac{d}{dx}(\sin 2x)}{\sqrt{1-(\sin 2x)^2}} = \frac{-2\cos 2x}{\sqrt{1-\sin^2 2x}} = \begin{cases} -2 & \text{if } n\pi - \frac{\pi}{4} < x < n\pi + \frac{\pi}{4} \\ 2 & \text{if } n\pi + \frac{\pi}{4} < x < n\pi + \frac{3\pi}{4} \end{cases}$ where n is any integer.

77. $\int_0^1 \frac{x^3}{1+x^8}\,dx = \int_0^1 \frac{x^3}{1+(x^4)^2}\,dx$. Let $u = x^4$, so $du = 4x^3\,dx$, $x = 0 \Rightarrow u = 0$, and $x = 1 \Rightarrow u = 1$. Then $\int_0^1 \frac{x^3}{1+x^8}\,dx = \frac{1}{4}\int_0^1 \frac{du}{1+u^2} = \frac{1}{4}\tan^{-1}u\Big|_0^1 = \frac{1}{4}\left(\frac{\pi}{4}\right) = \frac{\pi}{16}$.

78. Let $u = \sec^{-1}x$, so $du = \frac{dx}{|x|\sqrt{x^2-1}}$. Then $\int \frac{dx}{|x|\left(\sec^{-1}x\right)^3\sqrt{x^2-1}} = \int \frac{du}{u^3} = -\frac{1}{2u^2} + C = -\frac{1}{2\left(\sec^{-1}x\right)^2} + C$.

79. Let $u = \sin^{-1}x$, so $du = \frac{dx}{\sqrt{1-x^2}}$, $x = 0 \Rightarrow u = 0$, and $x = \frac{\sqrt{3}}{2} \Rightarrow u = \frac{\pi}{3}$. Then $\int_0^{\sqrt{3}/2} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/3} u\,du = \left.\frac{u^2}{2}\right|_0^{\pi/3} = \frac{\pi^2}{18}$.

6.7 Indeterminate Forms and l'Hôpital's Rule

19. $\lim_{x\to\infty}\frac{\ln\left(1+e^x\right)}{x^2} = \lim_{x\to\infty}\frac{e^x/\left(1+e^x\right)}{2x} = \lim_{x\to\infty}\frac{1}{\left(1/e^x+1\right)(2x)} = 0$

25. $\lim_{x\to 0}\frac{\sin x - x}{e^x - e^{-x} - 2x} = \lim_{x\to 0}\frac{\cos x - 1}{e^x + e^{-x} - 2} = \lim_{x\to 0}\frac{-\sin x}{e^x - e^{-x}} = -\lim_{x\to 0}\frac{\cos x}{e^x + e^{-x}} = -\frac{1}{2}$

30. $\lim_{x\to 0}\frac{\sin^{-1}x - x}{\tan^{-1}x - x} = \lim_{x\to 0}\frac{\frac{1}{\sqrt{1-x^2}} - 1}{\frac{1}{1+x^2} - 1} = \lim_{x\to 0}\frac{-\frac{1}{2}\left(1-x^2\right)^{-3/2}(-2x)}{-\frac{2x}{\left(1+x^2\right)^2}} = -\frac{1}{2}\lim_{x\to 0}\frac{\left(1+x^2\right)^2}{\left(1-x^2\right)^{3/2}} = -\frac{1}{2}$

52. $\lim_{x\to\infty} (1 - 1/x)^x$ is an indeterminate form of type 1^∞. Let $y = (1 - 1/x)^x$, so $\ln y = \ln(1 - 1/x)^x = x \ln(1 - 1/x)$. Then

$$\ln\left(\lim_{x\to\infty} y\right) = \lim_{x\to\infty} \ln y = \lim_{x\to\infty}\left[x \ln\left(1 - \frac{1}{x}\right)\right] = \lim_{x\to\infty} \frac{\ln\left(1 - \frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1/x^2}{1 - 1/x}}{-1/x^2} = \lim_{x\to\infty}\left(\frac{1}{1/x - 1}\right) = -1,$$

so $\lim_{x\to\infty} y = \lim_{x\to\infty} (1 - 1/x)^x = e^{-1} = 1/e$.