4.2 Concept Questions

1. See page 362.

2. If we put $u = x^3 + 1$, then $du = 3x^2\,dx \Rightarrow x^2\,dx = \frac{1}{3}\,du$, so $\int x^2 \cos\left(x^3+1\right) dx = \frac{1}{3}\int \cos u\,du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin\left(x^3+1\right) + C$. However, the substitution does not work for the second integral.

10. For $I = \int (1-3x)^{1.4}\,dx$, let $u = 1-3x \Rightarrow du = -3\,dx \Rightarrow dx = -\frac{1}{3}\,du$. Then $I = -\frac{1}{3}\int u^{1.4}\,du = -\frac{1}{3}\cdot\frac{1}{2.4}u^{2.4} + C = -\frac{1}{7.2}(1-3x)^{2.4} + C$.

12. For $I = \int x^2\left(2x^3-1\right)^{-4} dx$, let $u = 2x^3 - 1 \Rightarrow du = 6x^2\,dx \Rightarrow x^2\,dx = \frac{1}{6}\,du$. Then $I = \frac{1}{6}\int u^{-4}\,du = \frac{1}{6}\left(-\frac{1}{3}u^{-3}\right) + C = -\frac{1}{18}\left(2x^3-1\right)^{-3} + C$.

14. For $I = \int \left(x^2-1\right)\left(x^3-3x\right)^3 dx$, let $u = x^3 - 3x \Rightarrow du = \left(3x^2-3\right)dx \Rightarrow \left(x^2-1\right)dx = \frac{1}{3}\,du$. Then $I = \frac{1}{3}\int u^3\,du = \left(\frac{1}{3}\right)\left(\frac{1}{4}u^4\right) + C = \frac{1}{12}\left(x^3-3x\right)^4 + C$.

16. For $I = \int 2x\sqrt[3]{1-4x^2}\,dx$, let $u = 1 - 4x^2 \Rightarrow du = -8x\,dx \Rightarrow x\,dx = -\frac{1}{8}\,du$. Then $I = 2\left(-\frac{1}{8}\right)\int u^{1/3}\,du = -\frac{1}{4}\left(\frac{3}{4}u^{4/3}\right) + C = -\frac{3}{16}\sqrt[3]{(1-4x^2)^4} + C$.

22. For $I = \displaystyle\int \frac{\sqrt{1+u^{-1}}}{u^2}\,du$, let $v = 1 + u^{-1} \Rightarrow dv = -u^{-2}\,du$. Then $I = -\int v^{1/2}\,dv = -\frac{2}{3}v^{3/2} + C = -\frac{2}{3}\sqrt{(1+u^{-1})^3} + C$.

26. For $I = \int (2x+3)\sqrt{x-1}\,dx$, let $u = x - 1 \Rightarrow du = dx$ and $x = u + 1$. Then

$$I = \int [2(u+1)+3]\,u^{1/2}\,du = \int \left(2u^{3/2} + 5u^{1/2}\right) du = \frac{4}{5}u^{5/2} + \frac{10}{3}u^{3/2} + C$$

$$= \frac{2}{15}u^{3/2}(6u+25) + C = \frac{2}{15}(6x+19)\sqrt{(x-1)^3} + C.$$

28. For $I = \int x\sin x^2\,dx$, let $u = x^2 \Rightarrow du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}\,du$. Then $I = \frac{1}{2}\int \sin u\,du = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos x^2 + C$.

30. For $I = \int x^2\sec^2 x^3\,dx$, let $u = x^3 \Rightarrow du = 3x^2\,dx \Rightarrow x^2\,dx = \frac{1}{3}\,du$. Then $I = \frac{1}{3}\int \sec^2 u\,du = \frac{1}{3}\tan u + C = \frac{1}{3}\tan x^3 + C$.

34. For $I = \int \sqrt{\sin\theta}\cos\theta\,d\theta$, let $u = \sin\theta \Rightarrow du = \cos\theta\,d\theta$. Then $I = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}\sqrt{\sin^3\theta} + C$.

36. For $I = \displaystyle\int \frac{\sin x}{(1+\cos x)^3}\,dx$, let $u = 1 + \cos x \Rightarrow du = -\sin x\,dx$. Then $I = -\displaystyle\int \frac{du}{u^3} = -\int u^{-3}\,du = \frac{1}{2}u^{-2} + C = \frac{1}{2(1+\cos x)^2} + C$.

38. For $I = \displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$, let $u = \sqrt{x} = x^{1/2} \Rightarrow du = \frac{1}{2}x^{-1/2}\,dx = \frac{1}{2\sqrt{x}}\,dx \Rightarrow \frac{1}{\sqrt{x}}\,dx = 2\,du$. Then $I = 2\int \sin u\,du = -2\cos u + C = -2\cos\sqrt{x} + C$.

46. For $I = \int x^2 (1-x)^7 \, dx$, let $u = 1 - x \Rightarrow du = -dx$ and $x = 1 - u$. Then

$$I = -\int (1-u)^2 u^7 \, du = -\int \left(1 - 2u + u^2\right) u^7 \, du = -\int \left(u^9 - 2u^8 + u^7\right) du = -\tfrac{1}{10}u^{10} + \tfrac{2}{9}u^9 - \tfrac{1}{8}u^8 + C$$

$$= -\tfrac{1}{10}(1-x)^{10} + \tfrac{2}{9}(1-x)^9 - \tfrac{1}{8}(1-x)^8 + C.$$

48. For $I = \int \frac{x+1}{\left(\sqrt{x}-1\right)^{3/2}} \, dx$, let $u = \sqrt{x} - 1$, so $\sqrt{x} = u + 1 \Rightarrow x = u^2 + 2u + 1$ and $dx = 2(u+1)\, du$. Then

$$I = 2\int \frac{\left(u^2+2u+2\right)(u+1)}{u^{3/2}} \, du = 2\int \frac{u^3 + 3u^2 + 4u + 2}{u^{3/2}} \, du = 2\int \left(u^{3/2} + 3u^{1/2} + 4u^{-1/2} + 2u^{-3/2}\right) du$$

$$= 2\left(\tfrac{2}{5}u^{5/2} + 2u^{3/2} + 8u^{1/2} - 4u^{-1/2}\right) + C$$

$$= \tfrac{4}{5}\left(\sqrt{x}-1\right)^{5/2} + 4\left(\sqrt{x}-1\right)^{3/2} + 16\left(\sqrt{x}-1\right)^{1/2} - 8\left(\sqrt{x}-1\right)^{-1/2} + C.$$

52. $f(x) = \int f'(x)\, dx = \int \frac{x}{\left(2x^2+1\right)^{3/2}} \, dx$. Let $u = 2x^2 + 1 \Rightarrow du = 4x\, dx$. Then

$$f(x) = \frac{1}{4}\int \frac{du}{u^{3/2}} = \tfrac{1}{4}\int u^{-3/2} \, du = \tfrac{1}{4}\left(-2u^{-1/2}\right) + C = -\frac{1}{2\sqrt{2x^2+1}} + C.\ f(2) = -\tfrac{1}{6} \Rightarrow -\tfrac{1}{6} + C = -\tfrac{1}{6} \Rightarrow C = 0,$$

so $f(x) = -\dfrac{1}{2\sqrt{2x^2+1}}$.

61. True. Let $u = x^2$. Then $du = 2x\, dx \Rightarrow x\, dx = \frac{1}{2}\, du$, so $\int x f\left(x^2\right) dx = \frac{1}{2}\int f(u)\, du$.

62. False. Let $f(x) = x^2$, $a = 2$, and $b = 1$. Then $f(ax+b) = f(2x+1) = (2x+1)^2$, so
$\int f(ax+b)\, dx = \int (2x+1)^2 \, dx = \frac{1}{6}(2x+1)^3 + C_1$. But
$\int f(u)\, du = \int u^2 \, du = \frac{1}{3}u^3 + C_2 = \frac{1}{3}(2x+1)^3 + C_2 \neq \frac{1}{6}(2x+1)^3 + C_1$.

4.4 Concept Questions

1. See pages 387 and 388.
2. See pages 387 and 390-392.

14. $\lim_{n\to\infty} \sum_{k=1}^{n} 2c_k (1-c_k)^2 \, \Delta x = \int_0^3 2x(1-x)^2 \, dx$

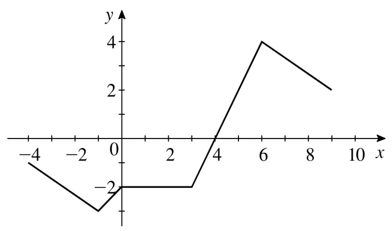
16. $\lim_{n\to\infty} \sum_{k=1}^{n} c_k \cos c_k \, \Delta x = \int_0^{\pi/2} x \cos x \, dx$

17. a. $\int_{-4}^{-1} f(x)\, dx = -\frac{1}{2}(1+3)3 = -6$

b. $\int_{-1}^{4} f(x)\, dx = -\left(\frac{5}{2} + 6 + 1\right) = -\frac{19}{2}$

c. $\int_{4}^{9} f(x)\, dx = \frac{1}{2}(6-4)4 + \frac{1}{2}(4+2)(9-6) = 4 + 9 = 13$

d. $\int_{-4}^{9} f(x)\, dx = \int_{-4}^{-1} f(x)\, dx + \int_{-1}^{4} f(x)\, dx + \int_{4}^{9} f(x)\, dx$
$= -6 - \frac{19}{2} + 13 = -\frac{5}{2}$

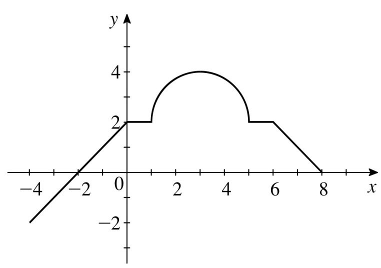


18. a. $\int_{-4}^{1} f(x)\, dx = -\frac{1}{2}(2)(2) + \frac{1}{2}(2)(2) + (1)(2) = 2$

b. $\int_{1}^{5} f(x)\, dx = (4)(2) + \frac{1}{2}\pi\left(2^2\right) = 8 + 2\pi$

c. $\int_{5}^{8} f(x)\, dx = (1)(2) + \frac{1}{2}(2)(2) = 4$

d. $\int_{-4}^{8} f(x)\, dx = \int_{-4}^{1} f(x)\, dx + \int_{1}^{5} f(x)\, dx + \int_{5}^{8} f(x)\, dx$
$= 2 + (8 + 2\pi) + 4 = 14 + 2\pi = 2(7+\pi)$



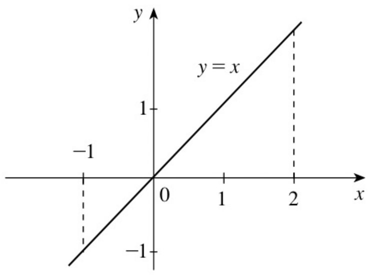
48. $-|f(x)| \le f(x) \le |f(x)|$ for all x in $[a, b]$, so by Property 5 of the definite integral,
$-\int_a^b |f(x)|\,dx = \int_a^b \left[-|f(x)|\right] dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx \Rightarrow \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$.

61. True. By the Sum Rule, $\int_a^b \left[f(x) + cg(x)\right] dx = \int_a^b f(x)\,dx + \int_a^b cg(x)\,dx = \int_a^b f(x)\,dx + c\int_a^b g(x)\,dx$, where we have used the Constant Multiple Rule.

62. False. Take $f(x) = g(x) = x$. Then $\int_0^1 f(x)\,g(x)\,dx = \int_0^1 x^2\,dx = \frac{1}{3}$ (see Example 8 in Section 4.3), but $\left[\int_0^1 f(x)\,dx\right]\left[\int_0^1 g(x)\,dx\right] = \left[\int_0^1 x\,dx\right]\left[\int_0^1 x\,dx\right] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \ne \frac{1}{3}$.

63. False. Take $a = 0$, $b = 1$, and $f(x) = 1$. Then $\int_a^b f(x)\,dx = \int_0^1 x\,dx = \frac{1}{2}$, but $x\int_a^b f(x)\,dx = x\int_0^1 1\,dx = x \ne \frac{1}{2}$.

64. False. Take $a = -1$, $b = 2$, and $f(x) = x$ on $[-1, 2]$. Then f is not positive on $[-1, 2]$. Indeed, $f\left(-\frac{1}{2}\right) = -\frac{1}{2} < 0$, but
$\int_a^b f(x)\,dx = \int_{-1}^2 x\,dx = \int_{-1}^0 x\,dx + \int_0^2 x\,dx = -\frac{1}{2} + 2 = \frac{3}{2} > 0$.



65. True. If f is decreasing on $[a, b]$, then the absolute maximum value of f on $[a, b]$ is $M = f(a)$ and the absolute minimum value of f on $[a, b]$ is $m = f(b)$. Therefore, $m \le f(x) \le M$ and Property 6 of the definite integral implies that $(b - a)\,f(b) \le \int_a^b f(x)\,dx \le (b - a)\,f(a)$.

66. True. We write $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^d f(x)\,dx + \int_d^b f(x)\,dx$. Since f is nonnegative, we have $\int_a^c f(x)\,dx \ge 0$ and $\int_c^d f(x)\,dx \ge 0$, so $\int_d^b f(x)\,dx \le \int_a^b f(x)\,dx$.