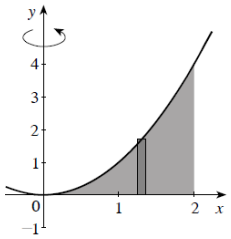
5.3 Volumes Using Cylindrical Shells

2. $V = 2\pi \int_0^2 xy\,dx = 2\pi \int_0^2 x\left(4-x^2\right)^{1/2}\,dx = 2\pi\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(4-x^2\right)^{3/2}\Big|_0^2 = \frac{16\pi}{3}$

7. $V = 2\pi \int_0^2 xy\,dx = 2\pi \int_0^2 x\left(x^2\right)dx$

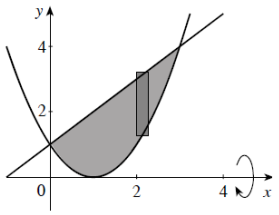
$= 2\pi \int_0^2 x^3\,dx = 2\pi\left(\frac{1}{4}x^4\right)\Big|_0^2 = 8\pi$



24. Solving $(x-1)^2 = x+1 \Leftrightarrow x^2-2x+1 = x+1 \Leftrightarrow$ $x^2-3x = x(x-3) = 0$ gives $x = 0$ or 3, so the points of intersection are $(0, 1)$ and $(3, 4)$. We use the method of disks (washers) to obtain

$V = \pi \int_0^3 \left[(x+1)^2-(x-1)^4\right]dx = \pi\left[\frac{1}{3}(x+1)^3-\frac{1}{5}(x-1)^5\right]_0^3$

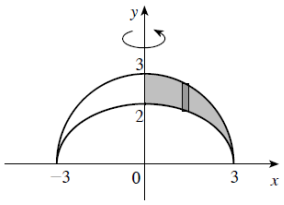
$= \frac{72\pi}{5}$



28. We solve $\sqrt{9-x^2} = \frac{2}{3}\sqrt{9-x^2} \Rightarrow 9-x^2 = \frac{4}{9}\left(9-x^2\right) \Leftrightarrow 5x^2 = 45$ $\Leftrightarrow x = \pm 3$, so the points of intersection are $(\pm 3, 0)$. We use the method of cylindrical shells to find

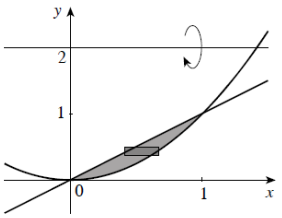
$V = 2\pi \int_0^3 x\left(\sqrt{9-x^2}-\frac{2}{3}\sqrt{9-x^2}\right)dx = \frac{2\pi}{3}\int_0^3 x\left(9-x^2\right)^{1/2}\,dx$

$= \frac{2\pi}{3}\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(9-x^2\right)^{3/2}\Big|_0^3 = 6\pi$



34. $V = 2\pi \int_0^1 (2-y)\left(\sqrt{y}-y\right)dy = 2\pi \int_0^1 \left(y^2-y^{3/2}-2y+2y^{1/2}\right)dy$

$= 2\pi\left(\frac{1}{3}y^3-\frac{2}{5}y^{5/2}-y^2+\frac{4}{3}y^{3/2}\right)\Big|_0^1 = \frac{8\pi}{15}$



5.4 Arc Length and Areas of Surfaces of Revolution

14. $y = \frac{x^3}{3}+\frac{1}{4x} \Rightarrow y' = x^2-\frac{1}{4x^2} = \frac{4x^4-1}{4x^2} \Rightarrow 1+(y')^2 = 1+\frac{16x^8-8x^4+1}{16x^4} = \frac{\left(4x^4+1\right)^2}{16x^4} \Rightarrow$

$L = \int_1^3 \sqrt{1+(y')^2}\,dx = \frac{1}{4}\int_1^3 \frac{4x^4+1}{x^2}\,dx = \frac{1}{4}\int_1^3 \left(4x^2+x^{-2}\right)dx = \frac{1}{4}\left(\frac{4x^3}{3}-\frac{1}{x}\right)\Big|_1^3 = \frac{53}{6}$

35. $2x+3y=6 \Rightarrow x=\dfrac{6-3y}{2} \Rightarrow x'=-\frac{3}{2} \Rightarrow 1+(x')^2=1+\frac{9}{4}=\frac{13}{4} \Rightarrow ds=\frac{\sqrt{13}}{2}\,dy$, so

$S=2\pi\int_{-2}^{1} x\,ds=2\pi\left(\frac{\sqrt{13}}{2}\right)\int_{-2}^{1}\left(3-\frac{3}{2}y\right)dy=\sqrt{13}\pi\left(3y-\frac{3}{4}y^2\right)\Big|_{-2}^{1}=\frac{45\sqrt{13}}{4}\pi.$

6.1 The Natural Logarithmic Function

28. $g(x)=\ln\left(x^2+4\right)^2=2\ln\left(x^2+4\right) \Rightarrow g'(x)=2\dfrac{d}{dx}\ln\left(x^2+4\right)=\dfrac{2(2x)}{x^2+4}=\dfrac{4x}{x^2+4}$

33. $y=x(\ln x)^2 \Rightarrow y'=x\dfrac{d}{dx}(\ln x)^2+(\ln x)^2\cdot\dfrac{d}{dx}(x)=x\cdot 2\ln x\cdot\dfrac{1}{x}+(\ln x)^2=(\ln x)^2+2\ln x$

48. $g(x)=\ln\sqrt{\dfrac{x\cos x}{(2x+1)^3}}=\frac{1}{2}\left[\ln(x\cos x)-3\ln(2x+1)\right] \Rightarrow$

$g'(x)=\dfrac{1}{2}\left(\dfrac{\cos x-x\sin x}{x\cos x}-\dfrac{6}{2x+1}\right)=\dfrac{(1-4x)\cos x-x(2x+1)\sin x}{2x(2x+1)\cos x}$

78. Let $u=1+\ln x$. Then $du=\dfrac{dx}{x}$, so $\displaystyle\int\frac{\sqrt{1+\ln x}}{x}\,dx=\int\sqrt{u}\,du=\frac{2}{3}u^{3/2}+C=\frac{2}{3}(1+\ln x)^{3/2}+C.$

81. $\displaystyle\int(\sec\theta+\cos\theta)\,d\theta=\int\frac{\sec\theta+\cos\theta}{1}\cdot\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\,d\theta=\int\frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\,d\theta+\int\cos\theta\,d\theta.$

To evaluate the first integral, we let $u=\sec\theta+\tan\theta$. Then $du=\left(\sec\theta\tan\theta+\sec^2\theta\right)d\theta$, so

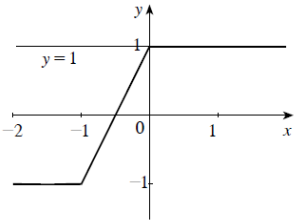
$\displaystyle\int(\sec\theta+\tan\theta)\,d\theta=\int\frac{du}{u}+\int\cos\theta\,d\theta=\ln|u|+\sin\theta+C=\ln|\sec\theta+\tan\theta|+\sin\theta+C.$

82. Let $u=1+\sin^2 x$. Then $du=2\sin x\cos x\,dx=\sin 2x\,dx$, so

$\displaystyle\int\frac{\sin 2x}{1+\sin^2 x}\,dx=\int\frac{du}{u}=\ln|u|+C=\ln\left(1+\sin^2 x\right)+C.$

6.2 Inverse Functions

18. $f(x)=|x+1|-|x|$ is not one-to-one. The horizontal line $y=1$ cuts the graph of f at infinitely many points.



26. By inspection $f(1)=1$, so $f^{-1}(1)=1$.

40. a. $f(g(x))=f\left(\frac{1}{2}+\sqrt{\frac{5}{4}-x}\right)=-\left(\frac{1}{2}+\sqrt{\frac{5}{4}-x}\right)^2+\left(\frac{1}{2}+\sqrt{\frac{5}{4}-x}\right)+1$

$=-\frac{1}{4}-\sqrt{\frac{5}{4}-x}-\frac{5}{4}+x+\frac{1}{2}+\sqrt{\frac{5}{4}-x}+1=x$

$g(f(x))=g\left(-x^2+x+1\right)=\frac{1}{2}+\sqrt{\frac{5}{4}+x^2-x-1}=\frac{1}{2}+\sqrt{x^2-x+\frac{1}{4}}$

$=\frac{1}{2}+\sqrt{\left(x-\frac{1}{2}\right)^2}=\frac{1}{2}+\left(x-\frac{1}{2}\right)=x$

b. $-x^2+x+1=\frac{1}{2}+\sqrt{\frac{5}{4}-x} \Rightarrow f(x)=g(x)$. The equation is found by solving $y=f(x)$ and $y=g(x)$ simultaneously, but f and g are inverse functions, so their graphs intersect at $y=x$. Setting $f(x)=x$ gives $-x^2+x+1=x \Rightarrow -x^2+1=0 \Rightarrow x=\pm 1$. We reject the negative root since y must be positive, so $x=1$.

56. $H'(x) = \dfrac{d}{dx}\left[g(g(x))\right] = g'(g(x))\,g'(x)$, so

$$H'(3) = g'(g(3))\,g'(3) = g'(4)\,g'(3) = \frac{1}{f'(g(4))}\cdot\frac{1}{f'(g(3))} = \frac{1}{f'(5)}\cdot\frac{1}{f'(4)} = \frac{1}{2}\cdot\frac{1}{\frac{1}{2}} = 1.$$

57. Observe that $f(2) = \displaystyle\int_2^2 \frac{dt}{\sqrt{1+t^3}} = 0$, showing that $(2, 0)$ lies on the graph of f. Next, observe that $f'(x) = \dfrac{d}{dx}\displaystyle\int_2^x \frac{dt}{\sqrt{1+t^3}} = \frac{1}{\sqrt{1+x^3}}$ (by the FTC, Part 1). Therefore,

$$\left(f^{-1}\right)'(0) = \frac{1}{f'\left(f^{-1}(0)\right)} = \frac{1}{f'(2)} = \sqrt{1+2^3} = 3.$$

60. a. $g'(x) = \dfrac{1}{f'(g(x))} = \left[f'(g(x))\right]^{-1}$, so by the Chain Rule,

$$g''(x) = -\left[f'(g(x))\right]^{-2}\frac{d}{dx}\left[f'(g(x))\right] = -\left[f'(g(x))\right]^{-2} f''(g(x))\,g'(x) = -\frac{f''(g(x))}{\left[f'(g(x))\right]^3}.$$

b. f is increasing on $(a, b) \Rightarrow f' > 0$ on (a, b). Also, the graph of f is concave upward on $(a, b) \Rightarrow f'' > 0$ on (a, b). Using the result of part a, we see that $g'' < 0$ on (a, b), and so the graph of g is concave downward on (a, b).