聯微作業解答 (3.2, 3.3)

3.2

• Verify that the function satisfies the hypotheses of Rolle's Theorem on the given interval, and find all values of c that satisfy the conclusion of the theorem.

5. $f(x) = x\sqrt{1-x^2}$; $[-1, 1]$

Sol. $f'(x) = (1-x^2)^{\frac{1}{2}} + x(\frac{1}{2}(1-x^2)^{-\frac{1}{2}})(-2x)$

$= (1-x^2)^{-\frac{1}{2}}((1-x^2) - x^2)$

$= \dfrac{1-2x^2}{\sqrt{1-x^2}}$

f is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$, $f(-1) = 0$, and $f(1) = 0$

by Rolle's Theorem, there exists a number c in $(-1, 1)$ such that $f'(c) = \dfrac{1-2c^2}{\sqrt{1-c^2}} = 0$.

In this case, $1 - 2c^2 = 0 \Rightarrow c = \pm\frac{\sqrt{2}}{2}$

• Verify that the function satisfies the hypotheses of Mean Value Theorem on the given interval, and find all values of c that satisfy the conclusion of the theorem.

10. $f(x) = x^3 - 2x$; $[-1, 2]$

Sol. $f'(x) = 3x^2 - 4x$

f is continuous on $[-1, 2]$ and differentiable on $(-1, 2)$,

by Mean Value Theorem, there exists a number c in $(-1, 2)$

such that

$\dfrac{f(2) - f(-1)}{2 - (-1)} = f'(c) \Rightarrow \dfrac{(8-8) - (-1-2)}{3} = 3c^2 - 4c$

$\Rightarrow 3c^2 - 4c - 1 = 0 \Rightarrow c = \dfrac{2 \pm \sqrt{7}}{3}$.

25. Prove that $f(x) = x^5 + 6x + 4$ has exactly one zero in $(-\infty, \infty)$.

pf. Since $f(x)$ is polynomial function, and so it is continuous and differentiable on $(-\infty, \infty)$.

Consider $f(-1) = -3 < 0$ and $f(0) = 4 > 0$, by Intermediate Value Theorem, there exists at least one number r_1 on $(-1, 0)$ such that $f(r_1) = 0$

Suppose f has another zero r_2 on $(-\infty, \infty)$, and $r_1 \neq r_2$.

Then $f(r_1) = f(r_2) = 0$, by Rolle's Theorem, there exists a number c in (r_1, r_2) such that $f'(c) = 0$. But f'$(x) = 5x^4 + 6 \geq 6$.

So no c exists. There is a contradiction.

Thus r_2 cannot exist. Hence f has exactly one zero.

26. Prove that the equation $x^7 + 6x5 + 2x - 6 = 0$ has exactly one real root.

pf. Let $f(x) = x^7 + 6x^5 + 2x - 6 = 0$. Since f is polynomial function, and so f is continuous and differential on $(-\infty, \infty)$.

Consider $f(-1) = -15 < 0$ and $f(1) = 3 > 0$, by Intermediate Value Theorem, there exists a number r_1 on $(-1, 1)$ such that $f(r_1) = 0$, and this show that f has at least one zero in $(-1, 1)$.

Suppose f has another zero r_2 on $(-\infty, \infty)$, and $r_1 \neq r_2$.

Then $f(r_1) = f(r_2) = 0$, by Rolle's Theorem, there exists a number c in (r_1, r_2) such that $f'(c) = 0$. But $f'(c) = 7c^6 + 30c^4 + 2 \geq 2$.

So no c exists. There is a contradiction.

Thus r_2 cannot exists. Hence f has exactly one zero, so the given equation has exactly one root.

34. Suppose that f and g are continuous on an interval $[a, b]$ and differentiable on the interval (a, b). Furthermore, suppose that $f(a) = g(a)$ and $f'(x) < g'(x)$ for $a < x < b$.

Prove that $f(x) < g(x)$ for a<x<b.

Hint: Apply the Mean Value Theorem to the function $h = f - g$.

pf. Let $h = f - g$. h is continuous on $[a, b]$ and differentiable on (a, b). $h(a) = (f - g)(a) = f(a) - g(a) = 0$. If $a < x < b$, by Mean Value Theorem, $\dfrac{h(x) - h(a)}{x - a} = h'(c)$, $c \in (a, x)$. Then $h'(x) = f'(x) - g'(x)$, and we have $\dfrac{h(x) - 0}{x - a} = f'(c) - g'(c) < 0$. Since $f'(x) < g'(x)$ for all $x \in (a, b)$. Multiplying both sides by $x - a$. $h(x) = f'(c) - g'(c) < 0$ since $x - a > 0 \Rightarrow f(x) < g(x)$ for all $x \in (a, b)$.

3.3

- (a) Find the intervals on which f is increasing or decreasing.

  (b) Find the relative maxima and relative minima of f.

10. $f(x) = -x^2 + 4x + 2$

Sol. $f'(x) = -2x + 4 = -2(x-2)$ is defined everywhere.

$f'(x) = 0 \Rightarrow x = 2$. The critical number of f is 2.

if $x > 2$, then$f'(x) < 0$

if $x < 2$, then$f'(x) > 0$

(a) f is increasing on $(-\infty, 2)$, and decreasing on $(2, \infty)$.

(b) f has relative maximum at $x = 2$ for $f(2) = 6$

18. $f(x) = x^{\frac{1}{3}} - x^{\frac{2}{3}}$

Sol. $f'(x) = \frac{1}{3}x^{-\frac{2}{3}} - \frac{2}{3}x^{\frac{2}{3}}$ is not defined at $x = 0$.

$f'(x) = 0 \Rightarrow x = \frac{1}{8}$. The critical number of f are 0 and 8.

if $x < 0$, then$f'(x) > 0$

if $0 < x < \frac{1}{8}$, then$f'(x) > 0$

if $x > \frac{1}{8}$, then$f'(x) < 0$

(a) f is increasing on $(-\infty, \frac{1}{8})$, and decreasing on $(\frac{1}{8}, \infty)$.

(b) f has relative maximum at $x = \frac{1}{8}$ for $f(\frac{1}{8}) = \frac{1}{4}$.

25. $f(x) = \dfrac{2x-3}{x^2-4}$

Sol. $f'(x) = \dfrac{2(x^2-4) - (2x-3)(2x)}{(x^2-4)^2} = \dfrac{-2(x^2-3x+4)}{(x+2)^2(x-2)^2}$ is not defined at $x = \pm 2$, and $f'(x)$ has no zero. Since $x = \pm 2$ are not in domain of f, there is no critical number of f, and $f'(x) < 0$ on its domain.

(a) f is decreasing on $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

(b) f has no relative extremum.

35. $f(x) = x\sin x + \cos x$, $0 < x < 2\pi$.

Sol. $f'(x) = \sin x + x\cos x - \sin x = x\cos x$ is defined everywhere.

$f'(x) = 0 \Rightarrow x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$. The critical number of f

are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

if $0 < x < \frac{\pi}{2}$, then$f'(x) > 0$

if $\frac{\pi}{2} < x < \frac{3\pi}{2}$, then$f'(x) < 0$

if $\frac{3\pi}{2} < x < 2\pi$, then $f'(x) > 0$

(a) f is increasing on $(0, \frac{\pi}{2})$, $(\frac{3\pi}{2}, 2\pi)$, and decreasing on $(\frac{\pi}{2}, \frac{3\pi}{2})$.

(b) f has relative maximum at $x = \frac{\pi}{2}$ for $f(\frac{\pi}{2}) = \frac{\pi}{2}$ and

f has relative minimum at $x = \frac{3\pi}{2}$ for $f(\frac{3\pi}{2}) = \frac{-3\pi}{2}$.

36. $f(x) = \dfrac{\sin x}{1 + \sin^2 x}$, $0 < x < 2\pi$.

Sol. $f'(x) = \dfrac{\cos x(1 + \sin^2 x) - \sin x(2\sin x\cos x)}{(1 + \sin^2 x)^2} = \dfrac{\cos^3 x}{(1 + \sin^2 x)^2}$

is defined everywhere.

$f'(x) = 0 \Rightarrow x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$. The critical number of f

are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

if $0 < x < \frac{\pi}{2}$, then$f'(x) > 0$

if $\frac{\pi}{2} < x < \frac{3\pi}{2}$, then$f'(x) < 0$

if $\frac{3\pi}{2} < x < 2\pi$, then $f'(x) > 0$

(a) f is increasing on $(0, \frac{\pi}{2})$, $(\frac{3\pi}{2}, 2\pi)$, and decreasing on $(\frac{\pi}{2}, \frac{3\pi}{2})$.

(b) f has relative maximum at $x = \frac{\pi}{2}$ for $f(\frac{\pi}{2}) = \frac{1}{2}$ and

f has relative minimum at $x = \frac{3\pi}{2}$ for $f(\frac{3\pi}{2}) = \frac{-1}{2}$.