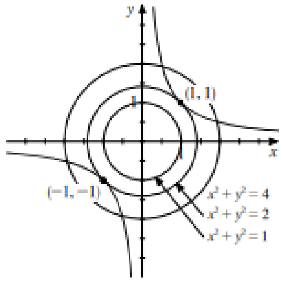
Week 17: 13.9: 3, 9, 11, 15, 17, 21

14.1: 本節沒有作業

3. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = y\mathbf{i} + x\mathbf{j}$, so $\nabla f = \lambda \nabla g \Rightarrow$ $2x\mathbf{i} + 2y\mathbf{j} = \lambda y\mathbf{i} + \lambda x\mathbf{j}$. Together with the constraint equation, we have

$$\left.\begin{aligned} 2x &= \lambda y \\ 2y &= \lambda x \\ xy &= 1 \end{aligned}\right\}$$

Solving the system, we find $\lambda = 2$, $x = -1$, and $y = -1$ or $\lambda = 2$, $x = 1$, and $y = 1$. The minimum value of f subject to $xy = 1$ is $f(-1,-1) = f(1,1) = 2$.



9. $f(x,y) = x^2 + xy + y^2 \Rightarrow \nabla f(x,y) = (2x+y)\,\mathbf{i} + (x+2y)\,\mathbf{j}$ and $g(x,y) = x^2 + y^2 - 8 \Rightarrow \nabla g(x,y) = 2x\mathbf{i} + 2y\mathbf{j}$,

so $\nabla f = \lambda \nabla g \Rightarrow (2x+y)\,\mathbf{i} + (x+2y)\,\mathbf{j} = 2\lambda x\mathbf{i} + 2\lambda y\mathbf{j}$, and we solve $\left.\begin{aligned} 2x + y &= 2\lambda x \\ x + 2y &= 2\lambda y \\ x^2 + y^2 &= 8 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x &= 2y(\lambda - 1) \\ y &= 2x(\lambda - 1) \end{aligned}\right\} \Rightarrow$

$y = \pm x$ (because $\lambda = 1 \Rightarrow x = y = 0$, violating the third equation). Substituting $y = \pm x$ into the third equation gives $x = \pm 2$ and $y = \pm 2$.

(x, y)	$(-2,-2)$	$(-2,2)$	$(2,-2)$	$(2,2)$
$f(x, y)$	12	4	4	12

From the table, we see that f has a minimum value of 4 and a maximum value of 12.

11. $f(x,y,z) = x + 2y + z \Rightarrow \nabla f(x,y,z) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $g(x,y,z) = x^2 + 4y^2 - z \Rightarrow \nabla g(x,y,z) = 2x\mathbf{i} + 8y\mathbf{j} - \mathbf{k}$, so

$\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} + 2\mathbf{j} + \mathbf{k} = 2\lambda x\mathbf{i} + 8\lambda y\mathbf{j} - \lambda\mathbf{k}$, and we solve $\left.\begin{aligned} 1 &= 2\lambda x \\ 2 &= 8\lambda y \\ 1 &= -\lambda \\ x^2 + 4y^2 - z &= 0 \end{aligned}\right\}$ We find $\lambda = -1$, $x = -\frac{1}{2}$,

$y = -\frac{1}{4}$, and $z = \frac{1}{2}$. We see that f has a minimum value of $f\left(-\frac{1}{2}, -\frac{1}{4}, \frac{1}{2}\right) = -\frac{1}{2}$.

15. $f(x,y,z) = xyz \Rightarrow \nabla f(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and $g(x,y,z) = x^2 + 2y^2 + \frac{1}{2}z^2 = 6 \Rightarrow$ $\nabla g(x,y,z) = 2x\mathbf{i} + 4y\mathbf{j} + z\mathbf{k}$, so $\nabla f = \lambda \nabla g \Rightarrow yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = 2\lambda x\mathbf{i} + 4\lambda y\mathbf{j} + \lambda z\mathbf{k}$, and we solve

$\left.\begin{aligned} yz &= 2\lambda x \\ xz &= 4\lambda y \\ xy &= \lambda z \\ x^2 + 2y^2 + \tfrac{1}{2}z^2 &= 6 \end{aligned}\right\}$ Solving the first three equations for λ gives $\lambda = \dfrac{yz}{2x} = \dfrac{xz}{4y} = \dfrac{xy}{z}$, so $x^2 = 2y^2$ and

$z^2 = 4y^2$. Substituting these into the last equation in the system gives $2y^2 + 2y^2 + \frac{1}{2}\left(4y^2\right) = 6 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$. Thus, $x = \pm\sqrt{2}$ and $z = \pm 2$. Evaluating $f(x,y,z)$ at $\left(\pm\sqrt{2}, -1, -2\right)$, $\left(\pm\sqrt{2}, -1, 2\right)$, $\left(\pm\sqrt{2}, 1, -2\right)$, and $\left(\pm\sqrt{2}, 1, 2\right)$, we see that f has a minimum value of $-2\sqrt{2}$ and a maximum value of $2\sqrt{2}$.

17. $f(x,y,z) = 2x + y \Rightarrow \nabla f(x,y,z) = 2\mathbf{i} + \mathbf{j}$, $g(x,y,z) = x + y + z - 1 \Rightarrow \nabla g(x,y,z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and $h(x,y,z) = y^2 + z^2 - 9 \Rightarrow \nabla h(x,y,z) = 2y\mathbf{j} + 2z\mathbf{k}$, so $\nabla f = \lambda \nabla g + \mu \nabla h$ along with the constraints $g(x,y,z) = 0$ and $h(x,y,z) = 0$ give the system

$$\left.\begin{aligned} 2 &= \lambda & (1) \\ 1 &= \lambda + 2\mu y & (2) \\ 0 &= \lambda + 2\mu z & (3) \\ x + y + z &= 1 & (4) \\ y^2 + z^2 &= 9 & (5) \end{aligned}\right\}$$

From (1), (2), and (3), we obtain $\mu = -\frac{1}{2y} = -\frac{1}{z}$ $\Rightarrow z = 2y$. Substituting this into (5) gives $y^2 + (2y)^2 = 5y^2 = 9 \Leftrightarrow y = \pm\frac{3\sqrt{5}}{5} \Rightarrow z = \pm\frac{6\sqrt{5}}{5}$. Then (4) gives $x = 1 - \left(\pm\frac{3\sqrt{5}}{5}\right) - \left(\pm\frac{6\sqrt{5}}{5}\right) = \frac{5 \mp 9\sqrt{5}}{5}$, so f has minimum value $f\left(\frac{5 - 9\sqrt{5}}{5}, \frac{3\sqrt{5}}{5}, \frac{6\sqrt{5}}{5}\right) = 2 - 3\sqrt{5}$ and maximum value $f\left(\frac{5 + 9\sqrt{5}}{5}, -\frac{3\sqrt{5}}{5}, -\frac{6\sqrt{5}}{5}\right) = 2 + 3\sqrt{5}$.

21. $$\left.\begin{aligned} f_x(x,y) &= \tfrac{\partial}{\partial x}\left(3x^2 + 2y^2 - 2x - 1\right) = 6x - 2 = 0 \\ f_y(x,y) &= \tfrac{\partial}{\partial y}\left(3x^2 + 2y^2 - 2x - 1\right) = 4y \quad\ = 0 \end{aligned}\right\}$$ $\Rightarrow x = \frac{1}{3}$ and $y = 0$, so f has the critical point $\left(\frac{1}{3}, 0\right)$ in the disk $D = \left\{(x,y) \mid x^2 + y^2 \le 9\right\}$. Next, we use the method of Lagrange to find the critical points of f on the boundary of D. We write $g(x,y) = x^2 + y^2 - 9 = 0$. Then $\nabla f(x,y) = (6x - 2)\mathbf{i} + 4y\mathbf{j}$ and $\nabla g(x,y) = 2x\mathbf{i} + 2y\mathbf{j}$. The equation $\nabla f = \lambda \nabla g$ and the constraint equation $g(x,y) = 0$ give the system

$$\left.\begin{aligned} 6x - 2 &= 2\lambda x & (1) \\ 4y &= 2\lambda y & (2) \\ x^2 + y^2 &= 9 & (3) \end{aligned}\right\}$$

Equation (2) gives $y = 0$ or $\lambda = 2$. If $y = 0$, then (3) gives $x = \pm 3$; if $\lambda = 2$, then (1) gives $x = 1$. Substituting this value of x into (3) gives $y = \pm 2\sqrt{2}$.

(x,y)	$\left(\frac{1}{3}, 0\right)$	$(-3, 0)$	$(3, 0)$	$\left(1, -2\sqrt{2}\right)$	$\left(1, 2\sqrt{2}\right)$
$f(x,y)$	$-\frac{4}{3}$	32	20	16	16

From the table, we see that f has a minimum value of $f\left(\frac{1}{3}, 0\right) = -\frac{4}{3}$ and a maximum value of $f(-3, 0) = 32$.