Week 16: 13.7: 7, 19, 25, 33

13.8: 9, 13, 34, 37

7. $F(x,y) = x^4 + 2x^2y^2 + y^4 - 9x^2 + 9y^2 \Rightarrow$
$\nabla F\left(\sqrt{5},-1\right) = \left[\left(4x^3+4xy^2-18x\right)\mathbf{i} + \left(4x^2y+4y^3+18y\right)\right]\mathbf{j}_{\left(\sqrt{5},-1\right)} = 6\sqrt{5}\mathbf{i} - 42\mathbf{j}$ is normal to the level curve $F(x,y) = x^4+2x^2y^2+y^4-9x^2+9y^2 = 0$ at $\left(\sqrt{5},-1\right)$, so the slope of the required normal line is $m = -\frac{42}{6\sqrt{5}} = -\frac{7\sqrt{5}}{5}$ and an equation of the normal line is $y-(-1) = -\frac{7\sqrt{5}}{5}\left(x-\sqrt{5}\right) \Leftrightarrow y = -\frac{7\sqrt{5}}{5}x + 6$. The slope of the required tangent line is $m = -\frac{1}{-\frac{7\sqrt{5}}{5}} = \frac{\sqrt{5}}{7}$, and an equation of the tangent line is $y-(-1) = \frac{\sqrt{5}}{7}\left(x-\sqrt{5}\right) \Leftrightarrow$ $y = \frac{\sqrt{5}}{7}x - \frac{12}{7}$.

19. $F(x,y,z) = xy+yz+xz-11 = 0 \Rightarrow \nabla F(1,2,3) = \left[(y+z)\mathbf{i}+(x+z)\mathbf{j}+(x+y)\mathbf{k}\right]_{(1,2,3)} = 5\mathbf{i}+4\mathbf{j}+3\mathbf{k}$, so an equation of the tangent plane at $(1,2,3)$ is $5(x-1)+4(y-2)+3(z-3) = 0 \Leftrightarrow 5x+4y+3z = 22$. Equations of the normal line passing through $(1,2,3)$ are $\frac{x-1}{5} = \frac{y-2}{4} = \frac{z-3}{3}$.

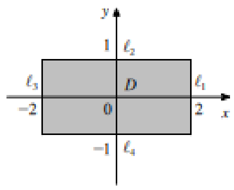
25. $F(x,y,z) = xe^y - z = 0 \Rightarrow \nabla F(2,0,2) = \left(e^y\mathbf{i}+xe^y\mathbf{j}-\mathbf{k}\right)\big|_{(2,0,2)} = \mathbf{i}+2\mathbf{j}-\mathbf{k}$, so an equation of the tangent plane at $(2,0,2)$ is $(x-2)+2y-(z-2) = 0 \Leftrightarrow x+2y-z = 0$. Equations of the normal line passing through $(2,0,2)$ are $x-2 = \frac{y}{2} = \frac{z-2}{-1}$.

33. $F(x,y,z) = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1 \Rightarrow \nabla F(x_0,y_0,z_0) = \left(\frac{2x}{a^2}\mathbf{i}+\frac{2y}{b^2}\mathbf{j}+\frac{2z}{c^2}\mathbf{k}\right)\Big|_{(x_0,y_0,z_0)} = \frac{2x_0}{a^2}\mathbf{i}+\frac{2y_0}{b^2}\mathbf{j}+\frac{2z_0}{c^2}\mathbf{k}$,
so an equation of the tangent plane at (x_0,y_0,z_0) is $\frac{2x_0}{a^2}(x-x_0)+\frac{2y_0}{b^2}(y-y_0)+\frac{2z_0}{c^2}(z-z_0) = 0 \Leftrightarrow$
$\frac{x_0x}{a^2}+\frac{y_0y}{b^2}+\frac{z_0z}{c^2}-\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}\right) = 0$. But (x_0,y_0,z_0) lies on the ellipsoid, so the expression in parentheses is equal to 1 and we have $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}+\frac{zz_0}{c^2} = 1$, as was to be shown.

9. $\left.\begin{array}{l} f_x(x,y) = \frac{\partial}{\partial x}\left(x^2+2y^2+x^2y+3\right) = 2x+2xy = 0 \\ f_y(x,y) = \frac{\partial}{\partial y}\left(x^2+2y^2+x^2y+3\right) = 4y+x^2 = 0 \end{array}\right\}$ The first equation gives $x = 0$ or $y = -1$. Substituting $x = 0$ into the second equation gives $y = 0$; substituting $y = -1$ into the second equation gives $x = \pm 2$. Thus, f has critical points $(0,0)$, $(-2,-1)$, and $(2,-1)$. Next, $D(x,y) = f_{xx}(x,y)\,f_{yy}(x,y) - f_{xy}^2(x,y) = (2+2y)(4)-(2x)^2 = -4x^2+8y+8$.
At $(0,0)$: $D(0,0) = 8 > 0$ and $f_{xx}(0,0) = 2 > 0$, so $(0,0)$ gives a relative minimum of f with value $f(0,0) = 3$.
At $(-2,-1)$: $D(-2,-1) = -4(-2)^2+8(-1)+8 = -16 < 0$, so $(-2,-1,5)$ is a saddle point of f.
At $(2,-1)$: $D(2,-1) = -4(2)^2+8(-1)+8 = -16 < 0$, so $(2,-1,5)$ is also a saddle point of f.

13. $\left.\begin{array}{l} f_x(x,y) = \frac{\partial}{\partial x}\left(x^2-6x-x\sqrt{y}+y\right) = 2x-6-\sqrt{y} = 0 \\ f_y(x,y) = \frac{\partial}{\partial y}\left(x^2-6x-xy^{1/2}+y\right) = -\frac{x}{2\sqrt{y}}+1 = 0 \end{array}\right\}$ From the first equation, we see that $\sqrt{y} = 2x-6$. Substituting this into the second equation gives $-x+2(2x-6) = 0 \Rightarrow x = 4$. Substituting this into the first equation gives $8-6 = \sqrt{y} \Rightarrow y = 4$, so the sole critical point of f is $(4,4)$. Next, $D(x,y) = f_{xx}(x,y)\,f_{yy}(x,y) - f_{xy}^2(x,y) = 2\left(\frac{x}{4y^{3/2}}\right) - \left(-\frac{1}{2\sqrt{y}}\right)^2 = \frac{x}{2y^{3/2}} - \frac{1}{4y}$. Since $D(4,4) = \frac{4}{2(8)} - \frac{1}{4(4)} = \frac{3}{16} > 0$ and $f_{xx}(4,4) = 2 > 0$, the point $(4,4)$ gives a relative minimum of f with value $f(4,4) = -12$.

34. $\left.\begin{array}{l} f_x(x,y) = \frac{\partial}{\partial x}\left(x^2+xy+y^2\right) = 2x+y = 0 \\ f_y(x,y) = \frac{\partial}{\partial y}\left(x^2+xy+y^2\right) = x+2y = 0 \end{array}\right\} \Rightarrow x = 0 \text{ and } y = 0,$



so $(0,0)$ is the only critical point of f in D.

On ℓ_1, $x = 2$ and $y = y$, so $g(y) = f(2,y) = 4+2y+y^2$ for $-1 \le y \le 1$. $g'(y) = 2+2y = 0 \Rightarrow y = -1$, so since $g(-1) = 3$ and $g(1) = 7$, we see that f has an absolute minimum value of 3 and an absolute maximum value of 7 on ℓ_1.

On ℓ_2, $x = x$ and $y = 1$, so $h(x) = f(x,1) = x^2+x+1$ for $-2 \le x \le 2$. $h'(x) = 2x+1 = 0 \Rightarrow x = -\frac{1}{2}$, so $-\frac{1}{2}$ is a critical number of h. Since $h(-2) = 3$, $h\left(-\frac{1}{2}\right) = \frac{3}{4}$, and $h(2) = 7$, we see that f has an absolute minimum value of $\frac{3}{4}$ and an absolute maximum value of 7 on ℓ_2.

On ℓ_3, $x = -2$ and $y = y$, so $s(y) = f(-2,y) = 4-2y+y^2$ for $-1 \le y \le 1$. $s'(y) = -2+2y = 0 \Rightarrow y = 1$, an endpoint. Since $s(-1) = 7$ and $s(1) = 3$, we see that f has an absolute minimum value of 3 and an absolute maximum value of 7 on ℓ_3.

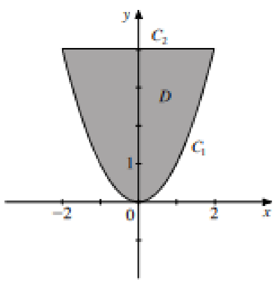
On ℓ_4, $x = x$ and $y = -1$, so $t(x) = f(x,-1) = x^2-x+1$ for $-2 \le x \le 2$. $g'(x) = 2x-1 = 0 \Rightarrow x = \frac{1}{2}$, so t has critical number $\frac{1}{2}$. Since $g(-2) = 7$, $g\left(\frac{1}{2}\right) = \frac{3}{4}$, and $g(2) = 3$, we see that f has an absolute minimum value of $\frac{3}{4}$ and an absolute maximum value of 7 on ℓ_4.

The extreme values of f on D and its boundaries are summarized below.

	Critical point	ℓ_1		ℓ_2		ℓ_3		ℓ_4	
(x,y)	$(0,0)$	$(2,-1)$	$(2,1)$	$\left(-\frac{1}{2},1\right)$	$(2,1)$	$(-2,-1)$	$(-2,1)$	$\left(\frac{1}{2},-1\right)$	$(-2,-1)$
Extreme value	0	3	7	$\frac{3}{4}$	7	7	3	$\frac{3}{4}$	7

We see that f has an absolute minimum value of 0 and an absolute maximum value of 7 on D.

37. $\left.\begin{array}{l} f_x(x,y) = \frac{\partial}{\partial x}\left(xy-x^2\right) = y-2x = 0 \\ f_y(x,y) = \frac{\partial}{\partial y}\left(xy-x^2\right) = x = 0 \end{array}\right\} \Rightarrow x = 0,\ y = 0, \text{ so } f \text{ has}$



no critical point in the interior of D.

On C_1, $y = x^2$, so $g(x) = f\left(x,x^2\right) = x^3-x^2$ for $-2 \le x \le 2$.

$g'(x) = 3x^2-2x = x(3x-2) = 0 \Rightarrow x = 0$ or $x = \frac{2}{3}$, so 0 and $\frac{2}{3}$ are critical numbers of g on $(-2,2)$.

x	-2	0	$\frac{2}{3}$	2
$g(x)$	-12	0	$-\frac{4}{27}$	4

From the table, we see that f has an absolute minimum value of -12 and an absolute maximum value of 4 on C_1.

On C_2, $x = x$ and $y = 4$, so $h(x) = f(x,4) = 4x-x^2$ for $-2 \le x \le 2$. $h'(x) = 4-2x = 0 \Rightarrow x = 2$, an endpoint. We find $h(-2) = -12$ and $g(2) = 4$, so f has an absolute minimum value of -12 and an absolute maximum value of 4 on C_2.

We conclude that f has an absolute minimum value of -12 and an absolute maximum value of 4 on D.