2.2 Concept Questions

1. a. The Power Rule: If n is any real number, then $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$.

b. The Constant Multiple Rule: If f is a differentiable function and c is a constant, then $\frac{a}{dx}\left[cf(x)\right] = cf'(x)$.

c. The Sum Rule: If f and g are differentiable functions, then $\frac{d}{dx}\left[f(x) + g(x)\right] = f'(x) + g'(x)$.

2. a. If $h(x) = 2f(x)$, then $h'(x) = 2f'(x)$ and $h'(2) = 2f'(2) = 2 \cdot 3 = 6$.

b. If $F(x) = 2f(x) - 4g(x)$, then $F'(x) = 2f'(x) - 4g'(x)$ and $F'(2) = 2f'(2) - 4g'(2) = 2(3) - 4(-2) = 6 + 8 = 14$.

8. $f'(u) = \frac{d}{du}\left(\frac{2}{\sqrt{u}}\right) = 2\frac{d}{du}\left(u^{-1/2}\right) = 2\left(-\frac{1}{2}\right)u^{-(1/2)-1} = -u^{-3/2} = -\frac{1}{u^{3/2}}$

13. $f'(r) = \frac{d}{dr}\left(\pi r^2 + 2\pi r\right) = 2\pi r + 2\pi$

18. $H(u) = (2u)^3 - 3u + 7 = 8u^3 - 3u + 7 \Rightarrow H'(u) = \frac{d}{du}\left(8u^3 - 3u + 7\right) = 24u^2 - 3 = 3\left(8u^2 - 1\right)$

24. $f'(x) = \frac{d}{dx}\left[-\frac{1}{3}\left(x^{-3} - x^6\right)\right] = -\frac{1}{3}\left(-3x^{-4} - 6x^5\right) = x^{-4} + 2x^5 = 2x^5 + \frac{1}{x^4}$

34. $f'(x) = \frac{d}{dx}\left(4x^{5/4} + 2x^{3/2} + x\right) = 5x^{1/4} + 3x^{1/2} + 1$

a. $f'(0) = 5(0) + 3(0) + 1 = 1$

b. $f'(16) = 5(16)^{1/4} + 3(16)^{1/2} + 1 = 10 + 12 + 1$

$= 23$

40. $g'(x) = x^2 - x - 1 = -1 \Rightarrow x^2 - x = x(x-1) = 0 \Rightarrow x = 0$ or 1. $g(0) = 1$ and $g(1) = \frac{1}{3} - \frac{1}{2} - 1 + 1 = -\frac{1}{6}$, so the points are $(0, 1)$ and $\left(1, -\frac{1}{6}\right)$.

42. $F'(s) = \frac{d}{ds}\left(2 + \frac{1}{s}\right) = -\frac{1}{s^2} = -\frac{1}{9} \Rightarrow s^2 = 9 \Rightarrow s = \pm 3$. $F(3) = \frac{2(3)+1}{3} = \frac{7}{3}$ and $F(-3) = \frac{2(-3)+1}{-3} = \frac{5}{3}$, so the points are $\left(3, \frac{7}{3}\right)$ and $\left(-3, \frac{5}{3}\right)$.

48. $y = \frac{1}{3}x^3 - 2x + 5 \Rightarrow \frac{ay}{dx} = x^2 - 2$. The slope of the given line is 2, so set $x^2 - 2 = 2 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$. The required points are $\left(-2, \frac{19}{3}\right)$ and $\left(2, \frac{11}{3}\right)$.

50. The line $y = 2x$ has slope 2. Also $y = x^2 + c \Rightarrow \frac{dy}{dx} = 2x$, so $2x = 2 \Rightarrow x = 1$. Therefore $y = 2$. Substituting into the second equation gives $2 = 1 + c$, so $c = 1$.

52. The parabola $y = x^2 - 6x + 11 = (x-3)^2 + 2$ has vertex $(3, 2)$, so the slope of the line passing through the vertex of the parabola and $(1, 0)$ is $\frac{2-0}{3-1} = 1$. The slope of the normal line is then $-1/m = -1$. The slope of the tangent line to the parabola at (x_0, y_0) is $\left.\frac{dy}{dx}\right|_{x=x_0} = [2x - 6]_{x=x_0} = 2x_0 - 6$. Since the slope of the required normal line is -1, it follows that $2x_0 - 6 = 1$ and $x_0 = \frac{7}{2}$. Since $f\left(\frac{7}{2}\right) = \left(\frac{7}{2}\right)^2 - 6\left(\frac{7}{2}\right) + 11 = \frac{9}{4}$, the point of intersection of the normal line and the parabola is $\left(\frac{7}{2}, \frac{9}{4}\right)$, and an equation of the normal line is $y - \frac{9}{4} = -1\left(x - \frac{7}{2}\right)$ or $y = -x + \frac{23}{4}$.

54. $\lim_{h\to 0} \dfrac{x^5-1}{x-1} = \lim_{h\to 0} \dfrac{(1+h)^5-1}{h} = f'(1)$, where $f(x) = x^5$. $f'(x) = 5x^4$, so $f'(1) = 5$.

56. Let $f(x) = x^{1/3}$. Then $f'(8) = \lim_{t\to 0} \dfrac{f(8+t)^{1/3} - f(8)}{t} = \lim_{t\to 0} \dfrac{(8+t)^{1/3}-2}{t}$. But $f'(8) = \frac{1}{3}x^{-2/3}\Big|_{x=8} = \dfrac{1}{3(8)^{2/3}} = \dfrac{1}{12}$, so the required limit is $\frac{1}{12}$.

58. We use the definition of the derivative to write

$$\lim_{h\to 0} \frac{\dfrac{1}{x+h} + \sqrt{x+h} - \dfrac{1}{x} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} + \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{d}{dx}\left(\frac{1}{x}\right) + \frac{d}{dx}\left(\sqrt{x}\right) = -\frac{1}{x^2} + \frac{1}{2\sqrt{x}}.$$

72. In order for f to be continuous at a, $\lim_{x\to a^-} f(x) = \lim_{x\to a^-} x^2 = a^2$ must be equal to $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} (Ax+B) = Aa + B$, that is, $aA + B = a^2$. In order for f to be differentiable at a, we must have $\lim_{h\to 0^-} \dfrac{f(a+h)-f(a)}{h} = \lim_{h\to 0^+} \dfrac{f(a+h)-f(a)}{h}$ or $2a = A$. Therefore, $A = 2a$ and $B = a^2 - 2a^2 = -a^2$.

75. False. $f'(x) = 2nx^{2n-1}$. For example, for $n = 3$, $f(x) = x^6 = x^{2\cdot 3}$ and $f'(x) = 6x^5$, whereas $2nx^{2(n-1)} = 2\cdot 3x^{2(3-1)} = 6x^4$.

76. False. $f(x) = 2^x$ is not a power function. The exponent in a power function must be a constant.

77. True by the Sum and Constant Multiple Rules.

78. False. Take $f(x) = x^2$. Then $g(x) = f\left(x^2\right) = \left(x^2\right)^2 = x^4$, so $g'(x) = 4x^3$. On the other hand, $f'(x) = 2x$, so $f'\left(x^2\right) = 2\left(x^2\right) = 2x^2$. Thus, $g'(x) \neq f'\left(x^2\right)$.

2.3 Concept Questions

1. a. The derivative of the product of two functions is equal to the first function times the derivative of the second function, plus the second function times the derivative of the first function.

b. The derivative of the quotient of two functions is equal to another quotient whose numerator is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, and whose denominator is given by the square of the denominator of the original quotient.

2. a. $h'(x) = f(x)g'(x) + f'(x)g(x)$, so $h'(1) = f(1)g'(1) + f'(1)g(1) = (3)(4) + (-1)(2) = 10$

b. $F'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$, so $F'(1) = \dfrac{g(1)f'(1) - f(1)g'(1)}{[g(1)]^2} = \dfrac{2(-1) - 3(4)}{2^2} = -\dfrac{7}{2}$.

18. $f'(t) = \left(1+t^{1/2}\right)\dfrac{d}{dt}\left(2t^2-3\right) + \left(2t^2-3\right)\dfrac{d}{dt}\left(1+t^{1/2}\right) = \left(1+t^{1/2}\right)(4t) + \left(2t^2-3\right)\left(\frac{1}{2}t^{-1/2}\right)$

$= 4t + 4t^{3/2} + t^{3/2} - \frac{3}{2}t^{-1/2} = 5t^{3/2} + 4t - \dfrac{3}{2\sqrt{t}}$

24. $f'(r) = \dfrac{d}{dr}\left[\dfrac{(2r+1)(r-3)}{3r+1}\right] = \dfrac{d}{dr}\left(\dfrac{2r^2-5r-3}{3r+1}\right) = \dfrac{(3r+1)\dfrac{d}{dr}\left(2r^2-5r-3\right) - \left(2r^2-5r-3\right)\dfrac{d}{dr}(3r+1)}{(3r+1)^2}$

$= \dfrac{(3r+1)(4r-5) - \left(2r^2-5r-3\right)(3)}{(3r+1)^2} = \dfrac{\left(12r^2-15r+4r-5\right) - \left(6r^2-15r-9\right)}{(3r+1)^2} = \dfrac{6r^2+4r+4}{(3r+1)^2}$

$= \dfrac{2\left(3r^2+2r+2\right)}{(3r+1)^2}$

34. $$f'(x) = \frac{(2x-1)\frac{d}{dx}(2x+1)-(2x+1)\frac{d}{dx}(2x-1)}{(2x-1)^2} = \frac{(2x-1)(2)-(2x+1)(2)}{(2x-1)^2} \Rightarrow$$

$$f'(2) = \frac{[2(2)-1](2)-[2(2)+1](2)}{[2(2)-1]^2} = \frac{6-10}{9} = -\frac{4}{9}$$

36. $$f'(x) = \frac{\left(x^4-2x^2-1\right)\frac{d}{dx}(x)-x\frac{d}{dx}\left(x^4-2x^2-1\right)}{\left(x^4-2x^2-1\right)^2} = \frac{\left(x^4-2x^2-1\right)(1)-x\left(4x^3-4x\right)}{\left(x^4-2x^2-1\right)^2} \Rightarrow$$

$$f'(-1) = \frac{\left[(-1)^4-2(-1)^2-1\right]-(-1)\left[\left(4(-1)^3-4(-1)\right)\right]}{\left[(-1)^4-2(-1)^2-1\right]^2} = \frac{1-2-1}{(1-2-1)^2} = -\frac{2}{4} = -\frac{1}{2}$$

48. $h'(x) = \left(x^2+1\right)g'(x)+g(x)(2x) \Rightarrow h'(1) = 2g'(1)+g(1)(2) = (2)(3)+(-2)(2) = 2$

50. $$h'(x) = \frac{[f(x)-g(x)]\frac{d}{dx}[f(x)g(x)]-f(x)g(x)\frac{d}{dx}[f(x)-g(x)]}{[f(x)-g(x)]^2}$$

$$= \frac{[f(x)-g(x)][f(x)g'(x)+g(x)f'(x)]-f(x)g(x)[f'(x)-g'(x)]}{[f(x)-g(x)]^2} \Rightarrow$$

$$h'(1) = \frac{[f(1)-g(1)][f(1)g'(1)+g(1)f'(1)]-f(1)g(1)[f'(1)-g'(1)]}{[f(1)-g(1)]^2}$$

$$= \frac{[2-(-2)][(2)(3)+(-2)(-1)]-(2)(-2)(-1-3)}{[2-(-2)]^2} = 1$$

52. Take $f(x) = (x+1)^2$. Then $f'(1) = \lim_{x\to1}\frac{f(x)-f(1)}{x-1} = \lim_{x\to1}\frac{(x+1)^2-4}{x-1}$. But $f'(x) = \frac{d}{dx}(x+1)^2 = \frac{d}{dx}\left(x^2+2x+1\right) = 2x+2$. Therefore, $\lim_{x\to1}\frac{(x+1)^2-4}{x-1} = f'(1) = (2x+2)|_{x=1} = 4$.

54. $f(x) = (2x)^4-(2x)^2+1 = 16x^4-4x^2+1 \Rightarrow f'(x) = 64x^3-8x \Rightarrow f''(x) = 192x^2-8$

58. $y = x^2\left(x+\frac{1}{x}\right) = x^3+x \Rightarrow y' = 3x^2+1 \Rightarrow y'' = 6x$

62. **a.** $f(x) = 8x^7-6x^5+4x^3-x \Rightarrow f'(x) = 56x^6-30x^4+12x^2-1 \Rightarrow f''(x) = 336x^5-120x^3+24x \Rightarrow$
$f'''(x) = 1680x^4-360x^2+24$, so $f'''(0) = 24$.

b. $y = x^{-1} \Rightarrow y' = -x^{-2} \Rightarrow y'' = 2x^{-3} \Rightarrow y''' = -6x^{-4}$, so $y'''|_{x=1} = -6(1)^{-4} = -6$.

73. False. Take $f(x) = x$ and $g(x) = x$. Then $f(x)g(x) = x^2$, so $\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}\left(x^2\right) = 2x \neq f'(x)g'(x) = 1$.

74. True. Using the Product Rule, $\frac{d}{dx}[xf(x)] = f(x)\frac{d}{dx}(x)+x\frac{d}{dx}[f(x)] = f(x)(1)+xf'(x)$.

75. True.
$\frac{d}{dx}[f(x)g'(x)-f'(x)g(x)] = f(x)g''(x)+f'(x)g'(x)-f'(x)g'(x)-f''(x)g(x) = f(x)g''(x)-f''(x)g(x)$.

76. True. If $P(x) = a_nx^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_0$ is a polynomial of degree n, then

$$\begin{aligned}
P'(x) &= na_nx^{n-1}+(n-1)a_{n-1}x^{n-2}+\cdots+3a_3x^2+2a_2x+a_1\\
P''(x) &= n(n-1)a_nx^{n-2}+(n-1)(n-2)a_{n-1}x^{n-3}+\cdots+6a_3x+2a_2\\
&\vdots\\
P^{(n-1)}(x) &= n(n-1)(n-2)\cdots(2)x\\
P^{(n)}(x) &= n(n-1)(n-2)\cdots(2)(1)\\
P^{(n+1)}(x) &= 0
\end{aligned}$$

77. True. $g(x) = [f(x)]^2 = f(x)f(x)$, so by the Product Rule, $g'(x) = f(x)f'(x)+f'(x)f(x) = 2f(x)f'(x)$.

78. True. $g(x) = [f(x)]^{-2} = \frac{1}{[f(x)]^2}$, so by the Quotient Rule,

$$g'(x) = \frac{[f(x)]^2\cdot\frac{d}{dx}(1)-1\cdot\frac{d}{dx}[f(x)]^2}{[f(x)]^4} = -\frac{2f(x)f'(x)}{[f(x)]^4} = -\frac{2f'(x)}{[f(x)]^3}.$$

2.5 Concept Questions

1. $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$, $\frac{d}{dx}(\tan x) = \sec^2 x$, $\frac{d}{dx}(\cot x) = -\csc^2 x$, $\frac{d}{dx}(\sec x) = \sec x \tan x$, $\frac{d}{dx}(\csc x) = -\csc x \cot x$

2. a. $\lim_{h\to 0} \frac{\cos(a+h) - \cos a}{h} = f'(a)$, where $f(x) = \cos x$. Since $f'(x) = -\sin x$, we see that $\lim_{h\to 0} \frac{\cos(a+h) - \cos a}{h} = f'(a) = -\sin a$.

b. $\lim_{h\to 0} \frac{\sec\left(\frac{\pi}{4}+h\right) - \sqrt{2}}{h} = \lim_{h\to 0} \frac{\sec\left(\frac{\pi}{4}+h\right) - \sec\frac{\pi}{4}}{h}$, where $f(x) = \sec x$. Since $f'(x) = \sec x \tan x$, we see that $\lim_{h\to 0} \frac{\sec\left(\frac{\pi}{4}+h\right) - \sqrt{2}}{h} = f'\left(\frac{\pi}{4}\right) = \sec\frac{\pi}{4}\tan\frac{\pi}{4} = \sqrt{2}(1) = \sqrt{2}$.

2. $g'(x) = \frac{d}{dx}(x + \tan x) = 1 + \sec^2 x$

4. $y' = \frac{d}{dx}\left(\sqrt{x}\sin x\right) = x^{1/2}\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}\left(x^{1/2}\right) = \sqrt{x}\cos x + \frac{\sin x}{2\sqrt{x}}$

8. $f'(t) = \frac{d}{dt}(\sec t \tan t) = \sec t\frac{d}{dt}(\tan t) + \tan t\frac{d}{dt}(\sec t) = (\sec t)\sec^2 t + \tan t(\sec t \tan t) = \sec t\left(\sec^2 t + \tan^2 t\right)$.

Alternative answers are $\sec t\left(1 + 2\tan^2 t\right)$ or $\sec t\left(2\sec^2 t - 1\right)$, using the identity $\sec^2 t = 1 + \tan^2 t$.

12. $y' = \frac{d}{d\theta}\left(\frac{\cos\theta}{1-\sin\theta}\right) = \frac{(1-\sin\theta)(-\sin\theta) - \cos\theta(-\cos\theta)}{(1-\sin\theta)^2} = \frac{-\sin\theta + \sin^2\theta + \cos^2\theta}{(1-\sin\theta)^2} = \frac{1-\sin\theta}{(1-\sin\theta)^2}$
$= \frac{1}{1-\sin\theta}$

16. $y = \cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 \Rightarrow$

$y' = \frac{d}{dx}\left(2\cos^2 x - 1\right) = 2\frac{d}{dx}[(\cos x)(\cos x)] - \frac{d}{dx}(1)$

$= 2[\cos x(-\sin x) + \cos x(-\sin x)] = -4\sin x\cos x = -2\sin 2x$

26. $h'(t) = \frac{d}{dt}\left[\left(t^2+1\right)\sin t\right] = \left(t^2+1\right)\frac{d}{dt}(\sin t) + \sin t\frac{d}{dt}\left(t^2+1\right) = \left(t^2+1\right)\cos t + 2t\sin t \Rightarrow$

$h''(t) = \frac{d}{dt}\left[\left(t^2+1\right)\cos t + 2t\sin t\right] = \left(t^2+1\right)\frac{d}{dt}(\cos t) + \cos t\frac{d}{dt}\left(t^2+1\right) + 2\left[t\frac{d}{dt}(\sin t) + \sin t\frac{d}{dt}(t)\right]$

$= -\left(t^2+1\right)\sin t + 2t\cos t + 2t\cos t + 2\sin t = 4t\cos t - t^2\sin t + \sin t$

28. $\frac{dw}{d\theta} = \frac{d}{d\theta}\left(\frac{\cos\theta}{\theta}\right) = \frac{\theta\frac{d}{d\theta}(\cos\theta) - \cos\theta\frac{d}{d\theta}(\theta)}{\theta^2} = -\frac{\theta\sin\theta + \cos\theta}{\theta^2} \Rightarrow$

$\frac{d^2w}{d\theta^2} = -\frac{\theta^2\frac{d}{d\theta}(\theta\sin\theta + \cos\theta) - (\theta\sin\theta + \cos\theta)\frac{d}{d\theta}\left(\theta^2\right)}{\theta^4}$

$= -\frac{\theta^2(\sin\theta + \theta\cos\theta - \sin\theta) - (\theta\sin\theta + \cos\theta)(2\theta)}{\theta^4} = \frac{2\cos\theta + 2\theta\sin\theta - \theta^2\cos\theta}{\theta^3}$

51. True. $f(x) = \frac{1-\sin^2 2x}{\cos^2 2x} = \frac{\cos^2 2x}{\cos^2 2x} = 1$, and so $f'(x) = 0$.

52. True.

$\frac{d}{dx}[\cos(x+h)] = \frac{d}{dx}(\cos x\cos h - \sin x\sin h) = \frac{d}{dx}(\cos x\cos h) - \frac{d}{dx}(\sin x\sin h)$

$= \cos h\frac{d}{dx}(\cos x) - \sin h\frac{d}{dx}(\sin x)$ (since h is a constant)

$= -\cos h\sin x - \sin h\cos x = -(\sin x\cos h + \cos x\sin h) = -\sin(x+h)$