2.6 Concept Questions

1. The derivative of $h(x) = g(f(x))$ is equal to the derivative of g evaluated at $f(x)$ times the derivative of f evaluated at x.

2. a. $g'(x) = \frac{d}{dx}(f(x))^n = n[f(x)]^{n-1} f'(x)$

b. $h'(x) = \frac{d}{dx}[\sec f(x)] = \sec f(x) \tan f(x) \cdot f'(x)$

3. a. $\dfrac{dP}{dt}$ measures the rate of change of the population P with respect to the temperature of the medium.

b. $\dfrac{dT}{dt}$ measures the rate of change of the temperature of the medium with respect to time.

c. $\dfrac{dP}{dt} = \dfrac{dP}{dT} \cdot \dfrac{dT}{dt} = f'(T)\, g'(t)$ measures the rate of change of the population with respect to time.

d. $(f \circ g)(t) = f(g(t)) = P$ gives the population of bacteria at any time t.

e. $f'(g(t))\, g'(t) = \dfrac{dP}{dt}$ (by the Chain Rule) gives the rate of change of the population with respect to time. See part c.

20. $f(x) = \left(x^2 + \sqrt{x}\right)^6 = \left(x^2 + x^{1/2}\right)^6 \Rightarrow f'(x) = 6\left(x^2 + x^{1/2}\right)^5 \left(2x + \frac{1}{2}x^{-1/2}\right) = 6\left(x^2 + \sqrt{x}\right)^5 \left(2x + \dfrac{1}{2\sqrt{x}}\right)$

22. $f(x) = \left(\dfrac{x+2}{x-3}\right)^{3/2} \Rightarrow$

$$f'(x) = \frac{3}{2}\left(\frac{x+2}{x-3}\right)^{1/2} \cdot \frac{(x-3)(1) - (x+2)(1)}{(x-3)^2} = \frac{3}{2}\left(\frac{x+2}{x-3}\right)^{1/2} \cdot \frac{-5}{(x-3)^2} = -\frac{15(x+2)^{1/2}}{2(x-3)^{5/2}} = -\frac{15\sqrt{x+2}}{2(x-3)^{5/2}}$$

30. $y = \cos\left(x^3\right) \Rightarrow y' = -\sin\left(x^3\right) \dfrac{d}{dx}\left(x^3\right) = -3x^2 \sin\left(x^3\right)$

32. $y = \cos\left(x^2 - 3x + 1\right) + \tan\left(\dfrac{2}{x}\right) \Rightarrow y' = -\sin\left(x^2 - 3x + 1\right) \dfrac{d}{dx}\left(x^2 - 3x + 1\right) +$

$$\sec^2\left(\frac{2}{x}\right) \frac{d}{dx}\left(2x^{-1}\right) = -(2x-3)\sin\left(x^2 - 3x + 1\right) - \frac{2}{x^2}\sec^2\left(\frac{2}{x}\right)$$

38. $g(x) = \tan^2\left(x^2 + x\right) \Rightarrow$

$$g'(x) = 2\tan\left(x^2 + x\right) \frac{d}{dx}\tan\left(x^2 + x\right) = 2\tan\left(x^2 + x\right)\sec^2\left(x^2 + x\right) \frac{d}{dx}\left(x^2 + x\right)$$
$$= 2(2x+1)\tan\left(x^2 + x\right)\sec^2\left(x^2 + x\right)$$

46. $g(t) = \sqrt{t + \tan 3t} \Rightarrow g'(t) = \dfrac{1}{2\sqrt{t + \tan 3t}} \cdot \dfrac{d}{dt}(t + \tan 3t) = \dfrac{1 + \sec^2 3t \dfrac{d}{dt}(3t)}{2\sqrt{t + \tan 3t}} = \dfrac{1 + 3\sec^2 3t}{2\sqrt{t + \tan 3t}}$

48. $y = \sec^3\left(\dfrac{\sqrt{x}}{1+x}\right) \Rightarrow$

$$\frac{dy}{dx} = 3\sec^2\left(\frac{\sqrt{x}}{1+x}\right)\sec\left(\frac{\sqrt{x}}{1+x}\right)\tan\left(\frac{\sqrt{x}}{1+x}\right)\frac{d}{dx}\left(\frac{\sqrt{x}}{1+x}\right)$$
$$= 3\sec^3\left(\frac{\sqrt{x}}{1+x}\right)\tan\left(\frac{\sqrt{x}}{1+x}\right) \cdot \frac{(1+x)\dfrac{1}{2\sqrt{x}} - \sqrt{x}(1)}{(1+x)^2} = 3\sec^3\left(\frac{\sqrt{x}}{1+x}\right)\tan\left(\frac{\sqrt{x}}{1+x}\right) \cdot \frac{1+x-2x}{2\sqrt{x}(1+x)^2}$$
$$= \frac{3(1-x)}{2\sqrt{x}(1+x)^2}\sec^3\left(\frac{\sqrt{x}}{1+x}\right)\tan\left(\frac{\sqrt{x}}{1+x}\right)$$

52. $y = x\tan^2(2x+3) \Rightarrow$

$$\frac{dy}{dx} = x\frac{d}{dx}\tan^2(2x+3) + \tan^2(2x+3)\frac{d}{dx}(x) = x[2\tan(2x+3)]\frac{d}{dx}[\tan(2x+3)] + \tan^2(2x+3)$$
$$= 2x\tan(2x+3)\sec^2(2x+3)\frac{d}{dx}(2x+3) + \tan^2(2x+3) = \tan(2x+3)\left[4x\sec^2(2x+3) + \tan(2x+3)\right]$$

54. $g(t) = \tan(\cos 2t) \Rightarrow g'(t) = \sec^2(\cos 2t)\,\dfrac{d}{dt}(\cos 2t) = \sec^2(\cos 2t)(-\sin 2t)\,\dfrac{d}{dt}(2t) = -2\sin 2t \sec^2(\cos 2t)$

56. $y = \sqrt{\sin(\cos 2x)} \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\sin(\cos 2x)}}\frac{d}{dx}[\sin(\cos 2x)] = \frac{1}{2\sqrt{\sin(\cos 2x)}}\cos(\cos 2x)\,\frac{d}{dx}(\cos 2x)$$
$$= \frac{\cos(\cos 2x)}{2\sqrt{\sin(\cos 2x)}}(-\sin 2x)\,\frac{d}{dx}(2x) = -\frac{\sin 2x\cos(\cos 2x)}{\sqrt{\sin(\cos 2x)}}$$

58. $g(x) = \dfrac{1}{(2x+1)^2} = (2x+1)^{-2} \Rightarrow g'(x) = -2(2x+1)^{-3}\,\dfrac{d}{dx}(2x+1) = -4(2x+1)^{-3} = -\dfrac{4}{(2x+1)^3} \Rightarrow$

$g''(x) = (-4)(-3)(2x+1)^{-4}\,\dfrac{d}{dx}(2x+1) = 24(2x+1)^{-4} = \dfrac{24}{(2x+1)^4}$

60. $y = x\sin\dfrac{1}{x} \Rightarrow$

$$\frac{dy}{dx} = x\frac{d}{dx}\sin\frac{1}{x} + \sin\frac{1}{x}\frac{d}{dx}x = x\cos\frac{1}{x}\frac{d}{dx}\frac{1}{x} + \sin\frac{1}{x} = x\cos\frac{1}{x}\left(-\frac{1}{x^2}\right) + \sin\frac{1}{x} = -x^{-1}\cos\frac{1}{x} + \sin\frac{1}{x} \quad \Rightarrow$$
$$\frac{d^2y}{dx^2} = -x^{-1}\frac{d}{dx}\cos\frac{1}{x} - \cos\frac{1}{x}\frac{d}{dx}x^{-1} + \frac{d}{dx}\sin\frac{1}{x} = -x^{-1}\left(-\sin\frac{1}{x}\right)\frac{d}{dx}\frac{1}{x} - \cos\frac{1}{x}\left(-\frac{1}{x^2}\right) + \cos\frac{1}{x}\frac{d}{dx}\frac{1}{x}$$
$$= x^{-1}\left(\sin\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + \frac{1}{x^2}\cos\frac{1}{x} - \frac{1}{x^2}\cos\frac{1}{x} = -\frac{1}{x^3}\sin\frac{1}{x}$$

99. $f(x) = \begin{cases} x^2\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Since

$$\tfrac{d}{dx}\left(x^2\sin\tfrac{1}{x}\right) = x^2\tfrac{d}{dx}\left(\sin\tfrac{1}{x}\right) + \sin\tfrac{1}{x}\tfrac{d}{dx}\left(x^2\right) = x^2\cos\tfrac{1}{x}\tfrac{d}{dx}\left(\tfrac{1}{x}\right) + 2x\sin\tfrac{1}{x} = x^2\left(-\tfrac{1}{x^2}\right)\cos\tfrac{1}{x} + 2x\sin\tfrac{1}{x}$$
$$= -\cos\tfrac{1}{x} + 2x\sin\tfrac{1}{x}$$

and $\displaystyle\lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2\sin\frac{1}{h}}{h} = \lim_{h\to 0} h\sin\frac{1}{h} = 0$ (by the Squeeze Theorem), we see that

$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Next,

$$\tfrac{d}{dx}\left(2x\sin\tfrac{1}{x} - \cos\tfrac{1}{x}\right) = 2x\tfrac{d}{dx}\left(\sin\tfrac{1}{x}\right) + \sin\tfrac{1}{x}\tfrac{d}{dx}(2x) + \sin\tfrac{1}{x}\tfrac{d}{dx}\left(\tfrac{1}{x}\right) = 2x\left(-\tfrac{1}{x^2}\right)\cos\tfrac{1}{x} + 2\sin\tfrac{1}{x} + \left(-\tfrac{1}{x^2}\right)\sin\tfrac{1}{x}$$
$$= 2\sin\tfrac{1}{x} - \tfrac{2}{x}\cos\tfrac{1}{x} - \tfrac{1}{x^2}\sin\tfrac{1}{x}$$

Thus, $f''(x) = \left(2 - \frac{1}{x^2}\right)\sin\frac{1}{x} - \frac{2}{x}\cos\frac{1}{x}$ for $x \neq 0$. Since $\frac{d}{dx}\sin\frac{1}{x}$ does not exist at $x = 0$, we conclude that $f''(x)$ does not exist at $x = 0$.

107. False. Let $f(x) = 2x+1$ and $g(x) = x^2$. Then $h(x) = g(f(x)) = (2x+1)^2$, so $h'(x) = 2(2x+1)(2) = 4(2x+1)$ and $h''(x) = 8$. But $g'(x) = 2x \Rightarrow g''(x) = 2$, so $g''(f(x))f''(x) = 2f''(x) = 2(0) = 0$. Therefore $h''(x) \neq g''(f(x))f''(x)$.

108. True. If $h(t) = f(a+bt) + f(a-bt)$, then $h'(t) = f'(a+bt)\frac{d}{dx}(a+bt) + f'(a-bt)\frac{d}{dt}(a-bt) = bf'(a+bt) - bf'(a-bt)$.

109. True. $\dfrac{d}{dx}f\left(\dfrac{1}{x}\right) = f'\left(\dfrac{1}{x}\right)\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = f'\left(\dfrac{1}{x}\right)\left(-\dfrac{1}{x^2}\right) = -\dfrac{f'\left(\frac{1}{x}\right)}{x^2}$.

110. True. $h(x) = [(f\circ f)(x)]^2 = [f(f(x))]^2 \Rightarrow h'(x) = 2[f(f(x))]\frac{d}{dx}f(f(x)) = 2[f(f(x))]f'(f(x))f'(x)$ or $h' = 2(f\circ f)(f'\circ f)f'$.

2.7 Concept Questions

1. a. We differentiate both sides of $F(x, y) = 0$ with respect to x, then solve for dy/dx.

b. The Chain Rule is used to differentiate any expression involving the dependent variable y.

2. Differentiating both sides of the equation $xg(y) + yf(x) = 0$ with respect to x gives
$x\frac{d}{dx}[g(y)] + g(y)\frac{d}{dx}(x) + y\frac{d}{dx}[f(x)] + f(x)\frac{d}{dx}(y) = 0 \Rightarrow x \cdot g'(y)\frac{dy}{dx} + g(y) \cdot 1 + y \cdot f'(x) + f(x) \cdot \frac{dy}{dx} = 0$
$\Rightarrow \frac{dy}{dx} = -\frac{g(y) + yf'(x)}{f(x) + xg'(y)}$.

8. $\frac{x^3}{y} + \frac{y^2}{x^2} = 3 \Rightarrow \frac{y(3x^2) - x^3y'}{y^2} + \frac{x^2(2yy') - y^2(2x)}{x^4} = 0 \Rightarrow 3x^6y - x^7y' + 2x^2y^3y' - 2xy^4 = 0 \Rightarrow$
$y' = \frac{y(2y^3 - 3x^5)}{x(2y^3 - x^5)}$

Alternate Solution: Start by multiplying both sides of the original equation by x^2y: $x^5 + y^3 = 3x^2y \Rightarrow$
$5x^4 + 3y^2y' = 6xy + 3x^2y' \Rightarrow y' = \frac{x(6y - 5x^3)}{3(y^2 - x^2)}$

10. $\frac{x+y}{x-y} = y^2 + 1 \Rightarrow \frac{(x-y)(1+y') - (x+y)(1-y')}{(x-y)^2} = 2yy' \Rightarrow$
$x + xy' - y - yy' - x + xy' - y + yy' = 2yy'(x^2 - 2xy + y^2) \Rightarrow 2xy' - 2yy'x^2 + 4xy^2y' - 2y^3y' = 2y \Rightarrow$
$y' = \frac{y}{x + 2xy^2 - x^2y - y^3}$

Alternate Solution: Start by multiplying both sides of the original equation by $x - y$:
$x + y = (x - y)(y^2 + 1) = xy^2 + x - y^3 - y \Rightarrow 1 + y' = y^2 + 2xyy' + 1 - 3y^2y' - y' \Rightarrow (3y^2 - 2xy + 2)y' = y^2$
$\Rightarrow y' = \frac{y^2}{3y^2 - 2xy + 2}$

12. $(2x^2 + 3y^2)^{5/2} = x \Rightarrow (2x^2 + 3y^2) = x^{2/5} \Rightarrow 4x + 6yy' = \frac{2}{5}x^{-3/5} \Rightarrow 6yy' = \frac{2}{5x^{3/5}} - 4x = \frac{2(1 - 10x^{8/5})}{5x^{3/5}} \Rightarrow$
$y' = \frac{1 - 10x^{8/5}}{15x^{3/5}y}$

16. $x + y^2 = \cos xy \Rightarrow 1 + 2yy' = (-\sin xy)(y + xy') \Rightarrow (2y + x\sin xy)y' = -y\sin xy - 1 \Rightarrow y' = -\frac{y\sin xy + 1}{2y + x\sin xy}$

22. $x^2y + y^3 = 2 \Rightarrow 2xy + x^2y' + 3y^2y' = 0 \Rightarrow y' = -\frac{2xy}{x^2 + 3y^2}$, so $y'|_{(-1,1)} = -\frac{2(-1)(1)}{1+3} = \frac{1}{2}$. An equation of the tangent line is $y - 1 = \frac{1}{2}(x + 1)$ or $y = \frac{1}{2}x + \frac{3}{2}$.

24. $y = \sin xy \Rightarrow y' = (\cos xy)(y + xy') \Rightarrow y' = \frac{y\cos xy}{1 - x\cos xy} \Rightarrow y'|_{(\pi/2,1)} = 0$. An equation of the tangent line is $y - 1 = 0(x - \frac{\pi}{2})$ or $y = 1$.

26. $x^{2/3} + y^{2/3} = 5 \Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$. At $(1, 8)$, $\frac{2}{3} + \frac{2}{3} \cdot 8^{-1/3}y' = 0 \Rightarrow y' = -2$.

28. $\tan(x + 2y) - \sin x = 1 \Rightarrow \sec^2(x + 2y)(1 + 2y') - \cos x = 0$. At $(0, \frac{\pi}{8})$, $2(1 + 2y') - 1 = 0 \Rightarrow y' = -\frac{1}{4}$.

30. $x^3 - y^3 = 8 \Rightarrow 3x^2 - 3y^2 y' = 0 \Rightarrow y' = \dfrac{x^2}{y^2}$. Differentiating both sides of the next-to-last expression yields

$$6x - 6y\left(y'\right)^2 - 3y^2 y'' = 0 \quad \Rightarrow \quad y'' = \frac{2\left[x - y\left(y'\right)^2\right]}{y^2} = \frac{2\left[x - y\left(x^2/y^2\right)^2\right]}{y^2} = \frac{2x\left(y^3 - x^3\right)}{y^5}$$

32. $\tan y - xy = 0 \Rightarrow \left(\sec^2 y\right) y' - y - xy' = 0 \Rightarrow y' = \dfrac{y}{\sec^2 y - x}$. Differentiating both sides of the next-to-last expression yields $\left(2\sec^2 y \tan y\right)\left(y'\right)^2 + \left(\sec^2 y\right) y'' - y' - y' - xy'' = 0 \Rightarrow$

$$y'' = \frac{2y'\left[1 - \left(\sec^2 y \tan y\right) y'\right]}{\sec^2 y - x} = \frac{2\left(\dfrac{y}{\sec^2 y - x}\right)\left(1 - \dfrac{y \sec^2 y \tan y}{\sec^2 y - x}\right)}{\sec^2 y - x} = \frac{2y\left(\sec^2 y - x - y\sec^2 y \tan y\right)}{\left(\sec^2 y - x\right)^3}$$

57. True. Differentiating both sides of the equation with respect to x, we have $\dfrac{d}{dx}\left[f(x)\,g(y)\right] = \dfrac{d}{dx}(0) \Rightarrow$ $f(x)\,g'(y)\dfrac{dy}{dx} + f'(x)\,g(y) = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{f'(x)\,g(y)}{f(x)\,g'(y)}$, provided $f(x) \neq 0$ and $g'(y) \neq 0$.

58. True. Differentiating both sides of the equation with respect to x, $\dfrac{d}{dx}\left[f(x) + g(y)\right] = \dfrac{d}{dx}(0) \Rightarrow f'(x) + g'(y)\dfrac{dy}{dx} = 0$ $\Rightarrow \dfrac{dy}{dx} = -\dfrac{f'(x)}{g'(y)}$.

Chapter 2 Review

Concept Review

1. a. $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ **b.** limit **c.** tangent line; $(a, f(a))$

d. $f(x)$; x; $(a, f(a))$ **e.** $y = f'(a)(x - a) + f(a)$

2. a. number; $f(x) = |x|$; 0 **b.** continuous; $f(x) = |x|$; 0; 0

3. a. 0 **b.** nx^{n-1}

4. $\dfrac{d}{dx}\left[c\left(f(x)\right)\right] = cf'(x)$; $\dfrac{d}{dx}\left[f(x) + g(x)\right] = f'(x) + g'(x)$; $\dfrac{d}{dx}\left[f(x)\,g(x)\right] = f(x)\,g'(x) + g(x)\,f'(x)$; $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)\,f'(x) - f(x)\,g'(x)}{\left[g(x)\right]^2}$

5. $\dfrac{dy}{dx}$

6. a. $f'(t)$; $f''(t)$; $f'''(t)$; $\left|f'(t)\right|$

b. > 0; < 0; 0

7. C'; R'; P'; $\overline{C}'$

8. x; $g'(f(x)) \cdot f'(x)$

9. a. $n\left[f(x)\right]^{n-1} \cdot f'(x)$

b. $\cos f(x) \cdot f'(x)$; $-\sin f(x) \cdot f'(x)$; $\sec^2 f(x) \cdot f'(x)$; $\sec f(x) \tan f(x) \cdot f'(x)$; $-\csc f(x) \cot f(x) \cdot f'(x)$; $-\csc^2 f(x) \cdot f'(x)$

10. both sides; $\dfrac{dy}{dx}$

11. y; $\dfrac{dy}{dt}$; a

12. $-\dfrac{y}{x}\dfrac{dy}{dt}$; $-\dfrac{y}{x}\dfrac{dx}{dt}$

13. a. $x_2 - x_1$ **b.** $f(x + \Delta x) - f(x)$

14. Δx; Δx; x; $f'(x)\,dx$