

6.5

$$33. f'(x) = \frac{d}{dx} \sin^{-1} 3x = \frac{1}{\sqrt{1-(3x)^2}} \cdot \frac{d}{dx} (3x) = \frac{3}{\sqrt{1-9x^2}}$$

$$35. f'(x) = \frac{d}{dx} \tan^{-1} x^2 = \frac{1}{1+(x^2)^2} \cdot \frac{d}{dx} (x^2) = \frac{2x}{1+x^4}$$

$$37. g'(t) = \frac{d}{dt} [t \tan^{-1} 3t] = \tan^{-1} 3t + t \cdot \frac{\frac{d}{dt} (3t)}{1+(3t)^2} = \tan^{-1} 3t + \frac{3t}{1+9t^2}$$

$$39. f'(u) = \frac{d}{du} \sec^{-1} 2u = \frac{\frac{d}{du} (2u)}{|2u| \sqrt{(2u)^2 - 1}} = \frac{1}{|u| \sqrt{4u^2 - 1}}$$

$$51. f'(x) = \frac{d}{dx} \sin^{-1} (e^{2x}) = \frac{1}{\sqrt{1-(e^{2x})^2}} \frac{d}{dx} e^{2x} = \frac{2e^{2x}}{\sqrt{1-e^{4x}}}$$

$$65. \int \frac{dx}{\sqrt{16-x^2}} = \int \frac{dx}{\sqrt{16[1-(x/4)^2]}} = \frac{1}{4} \int \frac{dx}{\sqrt{1-(x/4)^2}}. \text{ Let } u = \frac{1}{4}x, \text{ so } du = \frac{1}{4}dx. \text{ Then}$$

$$\int \frac{dx}{\sqrt{16-x^2}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1} \left(\frac{1}{4}x\right) + C.$$

$$67. \int_0^{1/2} \frac{dx}{1+4x^2} = \int_0^{1/2} \frac{dx}{1+(2x)^2}. \text{ Let } u = 2x, \text{ so } du = 2dx, x = 0 \Rightarrow u = 0, \text{ and } x = \frac{1}{2} \Rightarrow u = 1. \text{ Then}$$

$$\int_0^{1/2} \frac{dx}{1+4x^2} = \frac{1}{2} \int_0^1 \frac{du}{1+u^2} = \frac{\tan^{-1} u}{2} \Big|_0^1 = \frac{1}{2} \left(\frac{\pi}{4}\right) = \frac{\pi}{8}.$$

$$69. \text{ Let } u = \frac{1}{9}x^2, \text{ so } du = \frac{2}{9}x dx. \text{ Then}$$

$$\int \frac{dx}{x\sqrt{x^4-81}} = \frac{1}{9} \int \frac{x dx}{x^2 \sqrt{(x^2/9)^2 - 1}} = \frac{1}{9} \cdot \frac{9}{2} \int \frac{du}{9u \sqrt{u^2 - 1}} = \frac{1}{18} \sec^{-1} u + C = \frac{1}{18} \sec^{-1} \left(\frac{x^2}{9}\right) + C.$$

$$71. \int \frac{dt}{t\sqrt{t^6-16}} = \int \frac{dt}{t\sqrt{16[(t^6/16)-1]}} = \frac{1}{4} \int \frac{dt}{t\sqrt{(t^3/4)^2 - 1}}. \text{ Let } u = \frac{t^3}{4}, \text{ so } du = \frac{3t^2 dt}{4}. \text{ Then}$$

$$\int \frac{dt}{t\sqrt{t^6-16}} = \frac{1}{4} \int \frac{t^2 dt}{t^3 \sqrt{(t^3/4)^2 - 1}} = \frac{1}{4} \cdot \frac{4}{3} \cdot \frac{1}{4} \int \frac{du}{u \sqrt{u^2 - 1}} = \frac{1}{12} \sec^{-1} |u| + C = \frac{1}{12} \sec^{-1} \left|\frac{t^3}{4}\right| + C.$$

73. $\int_0^1 \frac{e^{2x}}{1+e^{4x}} dx = \int_0^1 \frac{e^{2x}}{1+(e^{2x})^2} dx$. Let $u = e^{2x}$, so $du = 2e^{2x} dx$, $x = 0 \Rightarrow u = 1$, and $x = 1 \Rightarrow u = e^2$. Then

$$\int_0^1 \frac{e^{2x}}{1+e^{4x}} dx = \frac{1}{2} \int_1^{e^2} \frac{du}{1+u^2} = \frac{1}{2} \tan^{-1} u \Big|_1^{e^2} = \frac{1}{2} \left(\tan^{-1} e^2 - \frac{\pi}{4} \right).$$

75. $\int \frac{\sin x}{\sqrt{4-\cos^2 x}} dx = \frac{1}{2} \int \frac{\sin x}{\sqrt{1-\left(\frac{\cos x}{2}\right)^2}} dx$. Let $u = \frac{1}{2} \cos x$, so $du = -\frac{1}{2} \sin x dx$. Then

$$\int \frac{\sin x}{\sqrt{4-\cos^2 x}} dx = \frac{1}{2} (-2) \int \frac{du}{\sqrt{1-u^2}} = -\sin^{-1} u + C = -\sin^{-1} \left(\frac{\cos x}{2} \right) + C.$$

77. $\int_0^1 \frac{x^3}{1+x^8} dx = \int_0^1 \frac{x^3}{1+(x^4)^2} dx$. Let $u = x^4$, so $du = 4x^3 dx$, $x = 0 \Rightarrow u = 0$, and $x = 1 \Rightarrow u = 1$. Then

$$\int_0^1 \frac{x^3}{1+x^8} dx = \frac{1}{4} \int_0^1 \frac{du}{1+u^2} = \frac{1}{4} \tan^{-1} u \Big|_0^1 = \frac{1}{4} \left(\frac{\pi}{4} \right) = \frac{\pi}{16}.$$

79. Let $u = \sin^{-1} x$, so $du = \frac{dx}{\sqrt{1-x^2}}$, $x = 0 \Rightarrow u = 0$, and $x = \frac{\sqrt{3}}{2} \Rightarrow u = \frac{\pi}{3}$. Then

$$\int_0^{\sqrt{3}/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \int_0^{\pi/3} u du = \frac{u^2}{2} \Big|_0^{\pi/3} = \frac{\pi^2}{18}.$$

81. Let $u = \tan^{-1} x$, so $du = \frac{dx}{1+x^2}$. Then $\int \frac{\tan^{-1} x}{1+x^2} dx = \int u du = \frac{1}{2} u^2 + C = \frac{1}{2} \left(\tan^{-1} x \right)^2 + C$.

83. Let $u = \sqrt{x}$, so $du = \frac{1}{2\sqrt{x}} dx$. Then $\int \frac{dx}{\sqrt{x}(4+x)} = \frac{1}{4} \int \frac{dx}{\sqrt{x}\left(1+\frac{1}{4}x\right)} = \frac{1}{4} \cdot 2 \int \frac{du}{1+\frac{1}{4}u^2} = \frac{1}{2} \int \frac{du}{1+\left(\frac{1}{2}u\right)^2}$. Next,

let $v = \frac{1}{2}u$, so $dv = \frac{1}{2} du$. Then $\int \frac{dx}{\sqrt{x}(4+x)} = \frac{1}{2} \cdot 2 \int \frac{dv}{1+v^2} = \tan^{-1} v + C = \tan^{-1} \left(\frac{1}{2}u \right) + C = \tan^{-1} \left(\frac{\sqrt{x}}{2} \right) + C$.

85. $\int \frac{dx}{4+(x-2)^2} = \frac{1}{4} \int \frac{dx}{1+\left(\frac{x-2}{2}\right)^2}$. Let $u = \frac{1}{2}(x-2)$, so $du = \frac{1}{2} dx$. Then

$$\int \frac{dx}{4+(x-2)^2} = \frac{1}{4} \cdot 2 \int \frac{du}{1+u^2} = \frac{1}{2} \tan^{-1} u + C = \frac{1}{2} \tan^{-1} \left(\frac{x-2}{2} \right) + C.$$

6.7

$$1. \lim_{x \rightarrow 1} \frac{x-1}{x^2-1} = \lim_{x \rightarrow 1} \frac{1}{2x} = \frac{1}{2}$$

$$3. \lim_{x \rightarrow 2} \frac{x^3-8}{x-2} = \lim_{x \rightarrow 2} \frac{3x^2}{1} = 12$$

$$5. \lim_{x \rightarrow 0} \frac{e^x-1}{x^2+x} = \lim_{x \rightarrow 0} \frac{e^x}{2x+1} = 1$$

$$7. \lim_{t \rightarrow \pi} \frac{\sin t}{\pi-t} = \lim_{t \rightarrow \pi} \frac{\cos t}{-1} = 1$$

$$11. \lim_{x \rightarrow \infty} \frac{x+\cos x}{2x+1} = \lim_{x \rightarrow \infty} \frac{1-\sin x}{2}. \text{ Observe that the limit on the right-hand side does not exist, so l'Hôpital's Rule does}$$

not apply. We evaluate the limit as follows: $\lim_{x \rightarrow \infty} \frac{x+\cos x}{2x+1} = \lim_{x \rightarrow \infty} \frac{1+\frac{\cos x}{x}}{2+\frac{1}{x}} = \frac{1}{2}$.

$$15. \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\ln x} = \lim_{x \rightarrow \infty} \frac{1/(2\sqrt{x})}{1/x} = \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{2} = \infty$$

$$19. \lim_{x \rightarrow \infty} \frac{\ln(1+e^x)}{x^2} = \lim_{x \rightarrow \infty} \frac{e^x/(1+e^x)}{2x} = \lim_{x \rightarrow \infty} \frac{1}{(1/e^x+1)(2x)} = 0$$

$$37. \lim_{x \rightarrow 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) = \lim_{x \rightarrow 1} \frac{x-1-\ln x}{(x-1)\ln x} = \lim_{x \rightarrow 1} \frac{1-\frac{1}{x}}{\ln x + \frac{x-1}{x}} = \lim_{x \rightarrow 1} \frac{x-1}{x \ln x + x-1} = \lim_{x \rightarrow 1} \frac{1}{\ln x + x \left(\frac{1}{x} \right) + 1} = \frac{1}{2}$$

$$41. \lim_{x \rightarrow \infty} \left(\frac{1}{x} e^{-x} \right) = \lim_{x \rightarrow \infty} \frac{1}{x e^x} = 0$$

$$42. \lim_{x \rightarrow 0^+} x^{\sin x} \text{ is an indeterminate form of type } 0^0, \text{ so let } y = x^{\sin x}. \text{ Then } \ln y = \ln x^{\sin x} = \sin x \ln x, \text{ so}$$

$$\begin{aligned} \ln \left(\lim_{x \rightarrow 0^+} y \right) &= \lim_{x \rightarrow 0^+} (\ln y) = \lim_{x \rightarrow 0^+} \frac{\ln x}{\csc x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-\csc x \cot x} = - \lim_{x \rightarrow 0^+} \frac{\sin^2 x}{x \cos x} = - \lim_{x \rightarrow 0^+} \left(\frac{\sin x}{x} \tan x \right) \\ &= 1 \cdot 0 = 0 \end{aligned}$$

$$\text{Thus, } \lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} x^{\sin x} = e^0 = 1.$$

$$52. \lim_{x \rightarrow \infty} (1-1/x)^x \text{ is an indeterminate form of type } 1^\infty. \text{ Let } y = (1-1/x)^x, \text{ so } \ln y = \ln(1-1/x)^x = x \ln(1-1/x). \text{ Then}$$

$$\ln \left(\lim_{x \rightarrow \infty} y \right) = \lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \left[x \ln \left(1 - \frac{1}{x} \right) \right] = \lim_{x \rightarrow \infty} \frac{\ln \left(1 - \frac{1}{x} \right)}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{1/x^2}{-1/x^2} = \lim_{x \rightarrow \infty} \left(\frac{1}{1/x-1} \right) = -1,$$

$$\text{so } \lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} (1-1/x)^x = e^{-1} = 1/e.$$