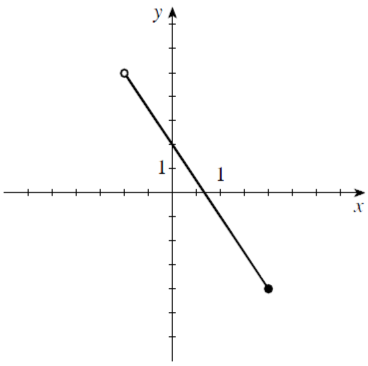
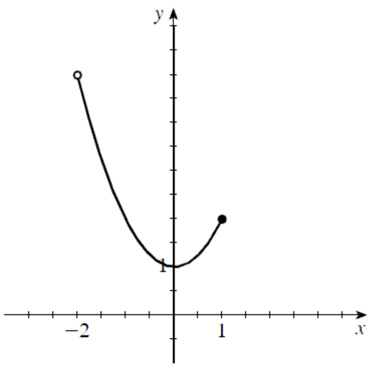
3.1 Extrema of Functions

8.



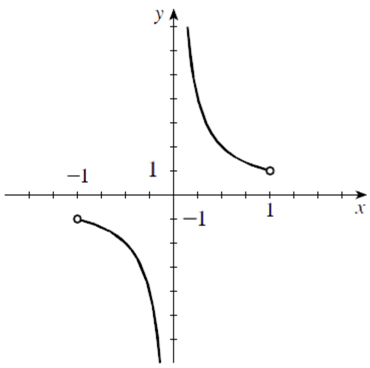
The absolute minimum value of g is -4, attained at $x = 2$.

12.



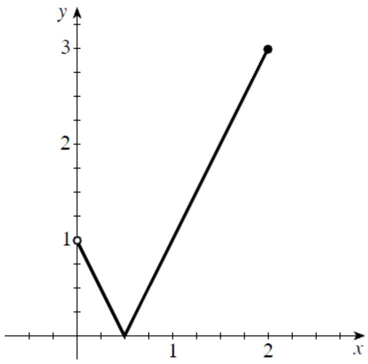
The absolute minimum value of h is 1, attained at $x = 0$.

16.



g has no absolute extremum.

18.



The absolute minimum value of g is 0, attained at $x = \frac{1}{2}$.

The absolute maximum value of g is 3, attained at $x = 2$.

46. $f'(t) = \frac{d}{dt}\left(-2t^3 + 3t^2 + 12t + 3\right) = -6t^2 + 6t + 12 = -6\left(t^2 - t - 2\right) = -6(t-2)(t+1)$, so -1 and 2 are critical numbers of f on the interval $(-2, 3)$. We calculate $f(-2) = 7$, $f(-1) = -4$, $f(2) = 23$, and $f(3) = 12$, so f has an absolute minimum value of -4 attained at $t = -1$ and an absolute maximum value of 23 attained at $t = 2$.

58. $g'(x) = \frac{d}{dx}(\cos x - \sin x) = -\sin x - \cos x$ is continuous everywhere and has zeros where $-\sin x - \cos x = 0 \Leftrightarrow \tan x = -1 \Leftrightarrow x = \frac{3\pi}{4}$ or $x = \frac{7\pi}{4}$ on the interval $(0, 2\pi)$. $g(0) = 1$, $g\left(\frac{3\pi}{4}\right) = -\sqrt{2}$, $g\left(\frac{7\pi}{4}\right) = \sqrt{2}$, and $g(2\pi) = 1$, so g has an absolute minimum value of $-\sqrt{2}$ attained at $x = \frac{3\pi}{4}$, and an absolute maximum value of $\sqrt{2}$ attained at $x = \frac{7\pi}{4}$.

60. $f'(x) = \frac{d}{dx}(x - \sin x) = 1 - \cos x$ is continuous everywhere and has zeros where $1 - \cos x = 0 \Leftrightarrow \cos x = 1 \Rightarrow x = 2n\pi$, n an integer. Since none of these lies on $(0, 2\pi)$, f has no critical number in that interval. $f(0) = 0$ and $f(2\pi) = 2\pi$, so f has an absolute minimum value of 0 attained at $x = 0$ and an absolute maximum value of 2π attained at $x = 2\pi$.

3.2 The Mean Value Theorem

24. $g(x) = x^4 - 2x^2 + x$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Furthermore, $g(0) = 0 = g(1)$. Therefore, by Rolle's Theorem, there exists at least one number c in $(0, 1)$ such that $g'(c) = 4c^3 - 4c + 1 = 0$. But $g'(x) = f(x)$, and so $f(c) = 0$, showing that f has at least one zero in $(0, 1)$.

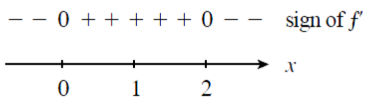
28. Let $f(x) = \sin x$. If $a = b$, then $|f(a) - f(b)| = |\sin a - \sin b| = 0$ and $|a - b| = 0$. So $|\sin a - \sin b| = |a - b|$. Next, we assume that $a < b$. The function f is continuous on $[a, b]$ and differentiable on (a, b). Using the Mean Value Theorem, we see that there exists a number c in (a, b) such that $\frac{f(b) - f(a)}{b - a} = \frac{\sin b - \sin a}{b - a} = f'(c) = \cos c$. Thus, $\sin b - \sin a = (b - a)\cos c \Rightarrow |\sin b - \sin a| = |b - a|\,|\cos c|$. But $|\cos c| \leq 1$ and so $|\sin b - \sin a| \leq |b - a|$ and, combined with the previous result, we see that the inequality holds for all real numbers a and b.

36. $f(x) = 2(x-1)(x-2)(x-3)(x-4)$ is a polynomial function, and thus continuous and differentiable on $(-\infty, \infty)$. Furthermore, $f(1) = f(2) = f(3) = f(4) = 0$. Therefore, by Rolle's Theorem, there exists at least one number c_1 in $(1, 2)$, at least one number c_2 in $(2, 3)$, and at least one number c_3 in $(3, 4)$ such that $f'(c_1) = f'(c_2) = f'(c_3) = 0$. Therefore, f' has at least three real zeros. On the other hand, f' is a polynomial of degree three and can have at most three real zeros. Therefore, f' has exactly three zeros.

37. Suppose f has two distinct fixed points c_1 and c_2 in I, with $c_1 < c_2$. Applying the Mean Value Theorem to f on the interval $[c_1, c_2]$ contained in I, we see that there exists a number c in (c_1, c_2) such that $f(c) = \dfrac{f(c_2) - f(c_1)}{c_2 - c_1} = \dfrac{c_2 - c_1}{c_2 - c_1} = 1$, contradicting the assumption that $f'(x) \neq 1$ for all x in I. This shows that f can have at most one fixed point in I.

3.3 Increasing and Decreasing Functions and the First Derivative Test

12. $f(x) = -x^3 + 3x^2 + 1 \Rightarrow f'(x) = -3x^2 + 6x = -3x(x-2)$ is continuous everywhere and has zeros at 0 and 2, the critical numbers of f. The sign diagram of f' is shown.

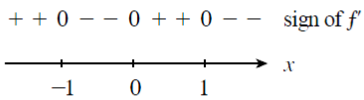


a. f is decreasing on $(-\infty, 0)$ and $(2, \infty)$ and increasing on $(0, 2)$.

b. f has a relative minimum of $f(0) = 1$ and a relative maximum of $f(2) = 5$.

16. $f(x) = -x^4 + 2x^2 + 1 \Rightarrow$

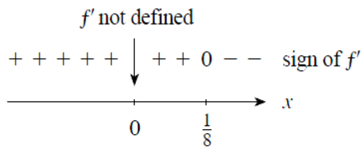
$f'(x) = -4x^3 + 4x = -4x\left(x^2 - 1\right) = -4x(x+1)(x-1)$ is continuous everywhere and has zeros at -1, 0, and 1, the critical numbers of f. The sign diagram of f' is shown.



a. f is increasing on $(-\infty, -1)$ and $(0, 1)$ and decreasing on $(-1, 0)$ and $(1, \infty)$.

b. f has relative maxima of $f(-1) = f(1) = 2$ and a relative minimum of $f(0) = 1$.

18. $f(x) = x^{1/3} - x^{2/3} \Rightarrow f'(x) = \frac{1}{3}x^{-2/3} - \frac{2}{3}x^{-1/3} = \frac{1}{3}x^{-2/3}\left(1 - 2x^{1/3}\right)$ is discontinuous at 0 and has a zero at $x = \frac{1}{8}$. The critical numbers of f are thus 0 and $\frac{1}{8}$. The sign diagram of f' is shown.

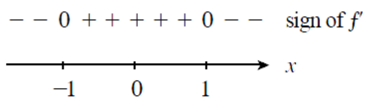


a. f is increasing on $\left(-\infty, \frac{1}{8}\right)$ and decreasing on $\left(\frac{1}{8}, \infty\right)$.

b. f has a relative maximum of $f\left(\frac{1}{8}\right) = \frac{1}{4}$.

24. $f(x) = \dfrac{x}{x^2 + 1} \Rightarrow$

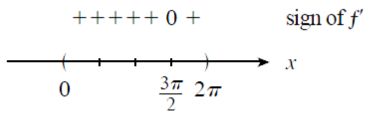
$f'(x) = \dfrac{(x^2+1)(1) - x(2x)}{\left(x^2+1\right)^2} = \dfrac{1 - x^2}{\left(x^2+1\right)^2} = \dfrac{(1-x)(1+x)}{\left(x^2+1\right)^2}$ is continuous everywhere and has zeros at ± 1, the critical numbers of f. The sign diagram of f' is shown.



a. f is decreasing on $(-\infty, -1)$ and $(1, \infty)$ and increasing on $(-1, 1)$.

b. f has a relative minimum of $f(-1) = -\frac{1}{2}$ and a relative maximum of $f(1) = \frac{1}{2}$.

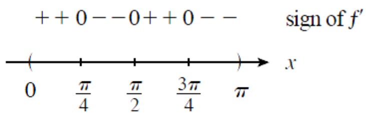
32. $f(x) = x - \cos x$, $0 < x < 2\pi \Rightarrow f'(x) = 1 + \sin x$ is continuous on $(0, 2\pi)$ and has zeros where $1 + \sin x = 0 \Leftrightarrow \sin x = -1 \Rightarrow x = \frac{3\pi}{2}$, a critical number of f. The sign diagram of f' is shown.



a. f is increasing on $(0, 2\pi)$.

b. f has no relative extremum.

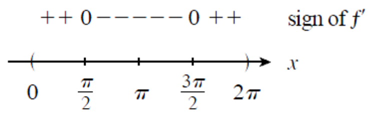
34. $f(x) = \sin^2 2x$, $0 < x < \pi \Rightarrow$
$f'(x) = 2(\sin 2x \cos 2x)(2) = 4 \sin 2x \cos 2x = 2 \sin 4x$ is continuous and has zeros where $\sin 4x = 0 \Rightarrow x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ in $(0, \pi)$. The sign diagram of f' is shown.



a. f is increasing on $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$ and decreasing on $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{4}, \pi\right)$.

b. f has relative maxima of $f\left(\frac{\pi}{4}\right) = f\left(\frac{3\pi}{4}\right) = 1$ and a relative minimum of $f\left(\frac{\pi}{2}\right) = 0$.

35. $f(x) = x \sin x + \cos x$, $0 < x < 2\pi \Rightarrow f'(x) = \sin x + x \cos x - \sin x = x \cos x$ is continuous everywhere and has zeros where $x \cos x = 0 \Rightarrow x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ in $(0, 2\pi)$. The sign diagram of f' is shown.



a. f is increasing on $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$ and decreasing on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

b. f has a relative maximum of $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$ and a relative minimum of $f\left(\frac{3\pi}{2}\right) = -\frac{3\pi}{2}$.

52. $f'(x) = \dfrac{d}{dx}(\cos x - ax + b) = -\sin x - a < 0 \Rightarrow \sin x > -a$. We require that the last inequality hold for all x on $(-\infty, \infty)$, and this is true provided $a > 1$.

57. $f(x) = ax^3 + 6x^2 + bx + 4 \Rightarrow f'(x) = 3ax^2 + 12x + b$. We require that $f'(-1) = 0 \Leftrightarrow 3a - 12 + b = 0$ and $f'(2) = 0 \Leftrightarrow 12a + 24 + b = 0$; solving these two equations simultaneously gives $a = -4$ and $b = 24$. Thus, $f(x) = -4x^3 + 6x^2 + 24x + 4$. Using the First Derivative Test, it can be shown that f has a relative minimum at -1 and a relative maximum at $x = 2$.