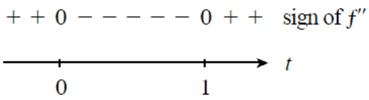
3.4 Concavity and Inflection Points

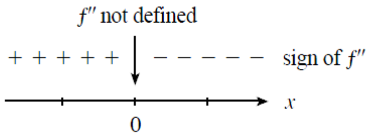
13. $f(t) = t^4 - 2t^3 \Rightarrow f'(t) = 4t^3 - 6t^2 \Rightarrow f''(t) = 12t^2 - 12t = 12t(t-1)$.

The sign diagram of f'' is shown at right. We see that f is concave upward on $(-\infty, 0)$ and $(1, \infty)$ and concave downward on $(0, 1)$. It has points of inflection at $(0, 0)$ and $(1, -1)$.



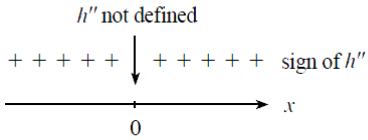
15. $f(x) = 1 + 3x^{1/3} \Rightarrow f'(x) = x^{-2/3} \Rightarrow f''(x) = -\frac{2}{3}x^{-5/3} = -\dfrac{2}{3x^{5/3}}$.

The sign diagram of f'' is shown at right. We see that f is concave upward on $(-\infty, 0)$ and concave downward on $(0, \infty)$. It has an inflection point at $(0, 1)$.



21. $h(x) = x^2 + x^{-2} \Rightarrow h'(x) = 2x - 2x^{-3} \Rightarrow h''(x) = 2 + 6x^{-4} = \dfrac{2\left(x^4 + 3\right)}{x^4}$.

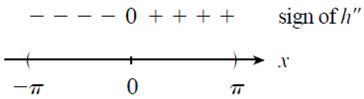
The sign diagram of h'' is shown at right. We see that h is concave upward on $(-\infty, 0)$ and $(0, \infty)$. It has no inflection point.



31. $h(x) = \dfrac{\sin x}{1 + \cos x}$, $-\pi < x < \pi \Rightarrow h'(x) = \dfrac{(1 + \cos x)\cos x - \sin x(-\sin x)}{(1 + \cos x)^2} = \dfrac{1}{1 + \cos x} \Rightarrow$

$h''(x) = \dfrac{d}{dx}(1 + \cos x)^{-1} = \dfrac{\sin x}{(1 + \cos x)^2}$.

The sign diagram of h'' is shown at right. We see that h is concave downward on $(-\pi, 0)$ and concave upward on $(0, \pi)$. It has an inflection point at $(0, 0)$.



37. $h(t) = \frac{1}{3}t^3 - 2t^2 - 5t - 10 \Rightarrow h'(t) = t^2 - 4t - 5 = (t-5)(t+1) = 0 \Rightarrow t = -1$ or 5, the critical numbers of h. $h''(t) = 2t - 4 = 2(t-2)$. We use the SDT: $h''(-1) = -6 < 0$, so h has a relative maximum of $h(-1) = -\frac{22}{3}$; and $h''(5) = 6 > 0$, so h has a relative minimum of $h(5) = -\frac{130}{3}$.

41. $f(t) = 2t + \dfrac{1}{t} \Rightarrow f'(t) = 2 - t^{-2} = \dfrac{2t^2 - 1}{t^2} = 0 \Rightarrow t = \pm\frac{\sqrt{2}}{2}$, the critical numbers of f. $f''(t) = \dfrac{2}{t^3}$, and we use the SDT: $f''\left(-\frac{\sqrt{2}}{2}\right) < 0$, so f has a relative maximum of $f\left(-\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}$; and $f''\left(\frac{\sqrt{2}}{2}\right) > 0$, so f has a relative minimum of $f\left(\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$.

45. $f(x) = \sin x + \cos x$, $0 < x < \frac{\pi}{2} \Rightarrow f'(x) = \cos x - \sin x = 0 \Leftrightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$ in $\left(0, \frac{\pi}{2}\right)$, so this is the only relevant critical number. $f''(x) = -\sin x - \cos x$, so using the SDT, we find that $f''\left(\frac{\pi}{4}\right) = -\sqrt{2} < 0$, implying that f has a relative maximum of $f\left(\frac{\pi}{4}\right) = \sqrt{2}$.

3.5 Limits Involving Infinity and Asymptotes

8. $\lim\limits_{t \to -3^+} \dfrac{t}{t+3} = -\infty$ since the numerator approaches -3 and the denominator approaches 0 through positive values as $t \to -3$ from the right.

12. $\lim\limits_{t \to 1} \dfrac{t^3}{(t^2-1)^2} = \infty$ since the numerator approaches 1 and the denominator approaches 0 through positive values as $t \to 1$.

14. $\lim\limits_{x \to -1^+} \left(\dfrac{1}{x} - \dfrac{1}{x+1}\right) = -\infty$. As $x \to -1$ from the right, the first term approaches -1 but the second term approaches ∞.

16. $\lim\limits_{x \to 0^+} \dfrac{1}{\sin x} = \infty$ since the numerator is positive and the denominator approaches 0 through positive values as $x \to 0$ from the right.

22. $\lim\limits_{x \to \infty} \dfrac{2x^2-1}{4x^2+1} = \lim\limits_{x \to \infty} \dfrac{2 - \frac{1}{x^2}}{4 + \frac{1}{x^2}} = \dfrac{1}{2}$

24. $\lim\limits_{x \to -\infty} \dfrac{2x^3+x^2+3}{x+1} = \lim\limits_{x \to -\infty} \dfrac{2x^2\left(1 + \frac{1}{2x} + \frac{3}{2x^3}\right)}{1 + \frac{1}{x}} = \infty$

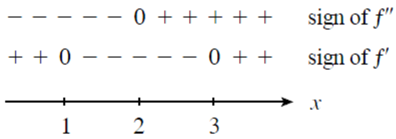
32. $\lim\limits_{t \to -\infty} \dfrac{2t^2}{\sqrt{t^4+t^2}} = \lim\limits_{t \to -\infty} \dfrac{2t^2}{\sqrt{t^4+t^2}} \cdot \dfrac{\frac{1}{t^2}}{\frac{1}{t^2}} = \lim\limits_{t \to -\infty} \dfrac{2}{\sqrt{1+\frac{1}{t^2}}} = 2$

50. $\lim\limits_{x \to \infty} \dfrac{x}{x+1} = 1$, so $y = 1$ is a horizontal asymptote. $\lim\limits_{x \to -1^+} \dfrac{x}{x+1} = -\infty$, so $x = -1$ is a vertical asymptote.

52. $\lim\limits_{t \to \infty} \dfrac{t^2}{t^2-4} = 1$, so $y = 1$ is a horizontal asymptote. $\lim\limits_{t \to -2^-} \dfrac{t^2}{(t+2)(t-2)} = \infty$ and $\lim\limits_{t \to 2^-} \dfrac{t^2}{(t+2)(t-2)} = -\infty$ so $t = \pm 2$ are vertical asymptotes.

54. $\lim\limits_{x \to \infty} \dfrac{2-x^2}{x^2+x} = -1$, so $y = -1$ is a horizontal asymptote. $\lim\limits_{x \to -1^-} \dfrac{2-x^2}{x(x+1)} = \infty$ and $\lim\limits_{x \to 0^+} \dfrac{2-x^2}{x(x+1)} = \infty$, so $x = -1$ and $x = 0$ are vertical asymptotes.

7. $f(x) = x^3 - 6x^2 + 9x + 2$

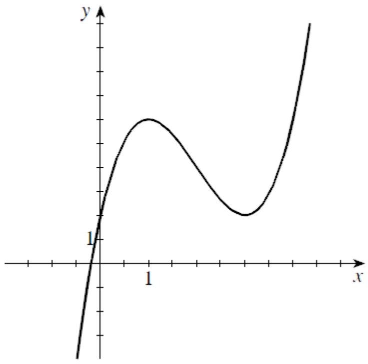


(1) The domain of f is $(-\infty, \infty)$. **(2)** The y-intercept is 2. The x-intercepts are not easily found, so we will not use this information. **(3)** There is no symmetry. **(4)** $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$. **(5)** f is a polynomial function, so there is no asymptote.

(6) $f'(x) = 3x^2 - 12x + 9 = 3\left(x^2 - 4x + 3\right) = 3(x-3)(x-1) = 0$ $\Leftrightarrow x = 1$ or 3, the critical numbers of f. From the sign diagram for f', we see that f is increasing on $(-\infty, 1)$ and $(3, \infty)$ and decreasing on $(1, 3)$.

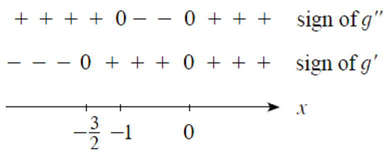
(7) From (6), we know that 1 and 3 are critical numbers of f. f' changes sign from positive to negative as we move across $x = 1$, so f has a relative maximum of $f(1) = 6$, and similarly, $f(3) = 2$ is a relative minimum.

(10)



(8) $f''(x) = 6x - 12 = 6(x-2) = 0 \Leftrightarrow x = 2$. The sign diagram for f'' shows that f is concave downward on $(-\infty, 2)$ and concave upward on $(2, \infty)$. **(9)** From (8), we see that f'' changes sign as we move across $x = 2$. Since $f(2) = 4$, f has an inflection point at $(2, 4)$.

11. $g(x) = x^4 + 2x^3 - 2$



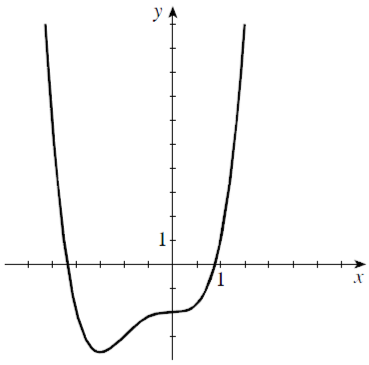
(1) The domain of g is $(-\infty, \infty)$. **(2)** The y-intercept is -2. **(3)** There is no symmetry. **(4)** $\lim_{x\to-\infty} g(x) = \lim_{x\to\infty} g(x) = \infty$.

(5) There is no asymptote. **(6)** $g'(x) = 4x^3 + 6x^2 = 2x^2(2x+3) = 0$ $\Leftrightarrow$ $x = 0$ or $-\frac{3}{2}$, the critical numbers of f. From the sign diagram for g', we see that g is decreasing on $\left(-\infty, -\frac{3}{2}\right)$ and increasing on $\left(-\frac{3}{2}, \infty\right)$.

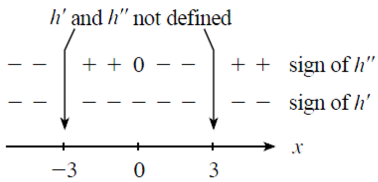
(7) g has a relative minimum at $\left(-\frac{3}{2}, -\frac{59}{16}\right)$.

(8) $g''(x) = 12x^2 + 12x = 12x(x+1) = 0 \Leftrightarrow x = 0$ or -1. From the sign diagram of g'', we see that g is concave upward on $(-\infty, -1)$ and $(0, \infty)$ and concave downward on $(-1, 0)$. **(9)** g has inflection points at $(-1, -3)$ and $(0, -2)$.

(10)



19. $h(x) = \dfrac{x}{x^2 - 9}$

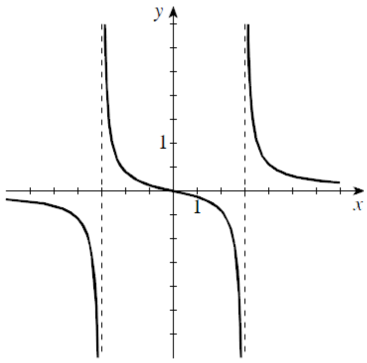


(1) The domain of h is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$. **(2)** The x- and y-intercepts are 0. **(3)** $h(-x) = \dfrac{-x}{(-x)^2 - 9} = -\dfrac{x}{x^2 - 9} = -h(x)$, so the graph of h is symmetric with respect to the origin.

(4) $\lim\limits_{x \to -\infty} h(x) = 0$ and $\lim\limits_{x \to \infty} h(x) = 0$. **(5)** From (4), we see that $y = 0$ is a horizontal asymptote. Since $\lim\limits_{x \to -3^-} h(x) = \lim\limits_{x \to 3^-} h(x) = -\infty$ and $\lim\limits_{x \to -3^+} h(x) = \lim\limits_{x \to 3^+} h(x) = \infty$, $x = \pm 3$ are vertical asymptotes.

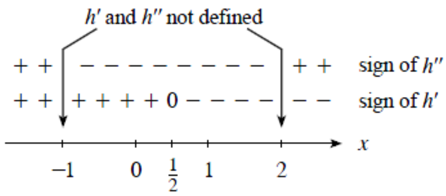
(6) $h'(x) = \dfrac{\left(x^2 - 9\right) - x(2x)}{\left(x^2 - 9\right)^2} = -\dfrac{x^2 + 9}{\left(x^2 - 9\right)^2}$. From the sign diagram for h' we see that h is decreasing on its domain. **(7)** f has no relative extremum.

(10)



(8) $h''(x) = -\dfrac{\left(x^2 - 9\right)^2 (2x) - \left(x^2 + 9\right) 2\left(x^2 - 9\right)(2x)}{\left(x^2 - 9\right)^4} = \dfrac{2x\left(x^2 + 27\right)}{\left(x^2 - 9\right)^3}$. From the sign diagram of h'', we see that h is concave downward on $(-\infty, -3)$ and $(0, 3)$ and concave upward on $(-3, 0)$ and $(3, \infty)$. **(9)** h has an inflection point at $(0, 0)$. Neither of ± 3 is in the domain of h.

23. $h(x) = \dfrac{1}{x^2 - x - 2}$

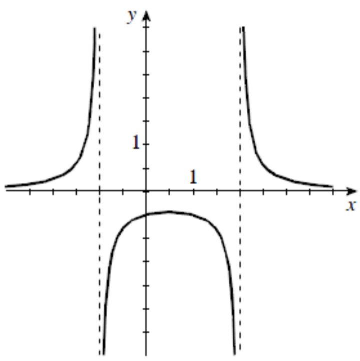


(1) Note that the denominator $x^2 - x - 2 = (x - 2)(x + 1) = 0 \Leftrightarrow x = -1$ or $x = 2$, so the domain of h is $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$.

(2) Setting $x = 0$ gives $-\frac{1}{2}$ as the y-intercept. **(3)** There is no symmetry.

(4) $\lim\limits_{x \to -\infty} h(x) = \lim\limits_{x \to \infty} h(x) = 0$ **(5)** The results of (4) show that $y = 0$ is a horizontal asymptote. Furthermore, the denominator is 0 at $x = -1$ or 2, where the numerator is not equal to 0. Therefore, $x = -1$ and $x = 2$ are vertical asymptotes.

(10)



(6) $h'(x) = \dfrac{d}{dx}\left(x^2 - x - 2\right)^{-1} = -\left(x^2 - x - 2\right)^{-2}(2x - 1)$

$$= \frac{1 - 2x}{\left(x^2 - x - 2\right)^2}$$

Setting $h'(x) = 0$ gives $x = \frac{1}{2}$. The sign diagram of h' shows that h is increasing on $(-\infty, -1)$ and $\left(-1, \frac{1}{2}\right)$ and decreasing on $\left(\frac{1}{2}, 2\right)$ and $(2, \infty)$. **(7)** The results of (5) show that $\left(\frac{1}{2}, -\frac{4}{9}\right)$ is a relative maximum.

(8) $h''(x) = \dfrac{\left(x^2 - x - 2\right)^2(-2) - (1 - 2x)\, 2\left(x^2 - x - 2\right)(2x - 1)}{\left(x^2 - x - 2\right)^4} = \dfrac{2\left(x^2 - x - 2\right)\left[-\left(x^2 - x - 2\right) + (2x - 1)^2\right]}{\left(x^2 - x - 2\right)^4}$

$$= \frac{6\left(x^2 - x + 1\right)}{\left(x^2 - x - 2\right)^3}$$

$h''(x)$ has no zero and is discontinuous at $x = -1$ and $x = 2$. The sign diagram of h'' shows that the graph of h is concave upward on $(-\infty, -1)$ and $(2, \infty)$ and concave downward on $(-1, 2)$. **(9)** There is no inflection point.

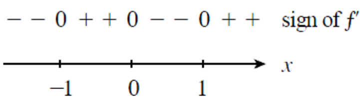
3. $h(x) = x^3 - 6x^2$ is continuous on $[2, 5]$. $h'(x) = 3x^2 - 12x = 3x(x-4) = 0 \Leftrightarrow x = 0$ or $x = 4$, so 4 is the only critical number of h in $(2, 5)$. $h(2) = -16$, $h(4) = -32$, and $h(5) = -25$, so the absolute maximum value is -16 attained at 2, and the absolute minimum value is -32 attained at 4.

5. $f(x) = 4x - \dfrac{1}{x^2}$ is continuous on $[1, 3]$. $f'(x) = 4 + \dfrac{2}{x^3} = \dfrac{4x^3+2}{x^3} = \dfrac{2\left(2x^3+1\right)}{x^3} = 0 \Rightarrow x = -\dfrac{1}{\sqrt[3]{2}}$, so f has no critical number in $(1, 3)$. $f(1) = 3$ and $f(3) = \frac{107}{9}$, so the absolute maximum value is $\frac{107}{9}$ attained at 3, and the absolute minimum value is 3 attained at 1.

9. $f(x) = \cos x - \sin x$ is continuous on $[0, 2\pi]$. $f'(x) = -\sin x - \cos x = 0 \Leftrightarrow \tan x = -1 \Rightarrow x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$ in $(0, 2\pi)$. $f(0) = 1$, $f\left(\frac{3\pi}{4}\right) = -\sqrt{2}$, $f\left(\frac{7\pi}{4}\right) = \sqrt{2}$, and $f(2\pi) = 1$, so the absolute maximum value is $\sqrt{2}$ attained at $\frac{7\pi}{4}$, and the absolute minimum value is $-\sqrt{2}$ attained at $\frac{3\pi}{4}$.

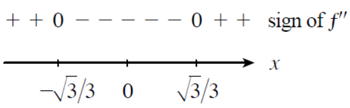
23. a. $f(x) = x^4 - 2x^2 \Rightarrow$

$f'(x) = 4x^3 - 4x = 4x\left(x^2-1\right) = 4x(x+1)(x-1)$. The sign diagram of f' shows that f is decreasing on $(-\infty, -1)$ and $(0, 1)$ and increasing on $(-1, 0)$ and $(1, \infty)$.



b. The results of part a and the First Derivative Test show that $(-1, -1)$ and $(1, -1)$ are relative minima and $(0, 0)$ is a relative maximum.

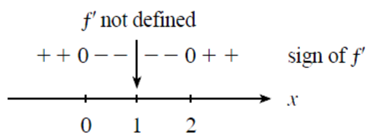
c. $f''(x) = 12x^2 - 4 = 4\left(3x^2-1\right) = 0 \Rightarrow x = \pm\frac{\sqrt{3}}{3}$. The sign diagram shows that f is concave upward on $\left(-\infty, -\frac{\sqrt{3}}{3}\right) \cup \left(\frac{\sqrt{3}}{3}, \infty\right)$ and concave downward on $\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$.



d. The results of part c show that $\left(-\frac{\sqrt{3}}{3}, -\frac{5}{9}\right)$ and $\left(\frac{\sqrt{3}}{3}, -\frac{5}{9}\right)$ are inflection points.

25. a. $f(x) = \dfrac{x^2}{x-1} \Rightarrow$

$f'(x) = \dfrac{(x-1)(2x) - x^2(1)}{(x-1)^2} = \dfrac{x^2-2x}{(x-1)^2} = \dfrac{x(x-2)}{(x-1)^2}$. The sign diagram of f' shows that f is increasing on $(-\infty, 0)$ and $(2, \infty)$ and decreasing on $(0, 1)$ and $(1, 2)$.



b. The results of part a show that $(0, 0)$ is a relative maximum and $(2, 4)$ is a relative minimum.

c. $f''(x) = \dfrac{(x-1)^2(2x-2) - x(x-2)\,2(x-1)}{(x-1)^4} = \dfrac{2(x-1)\left[(x-1)^2 - x(x-2)\right]}{(x-1)^4} = \dfrac{2}{(x-1)^3}$. Since $f''(x) < 0$ if $x < 1$ and $f''(x) > 0$ if $x > 1$, we see that f is concave downward on $(-\infty, 1)$ and concave upward on $(1, \infty)$.

d. Since $x = 1$ is not in the domain of f, there is no inflection point.

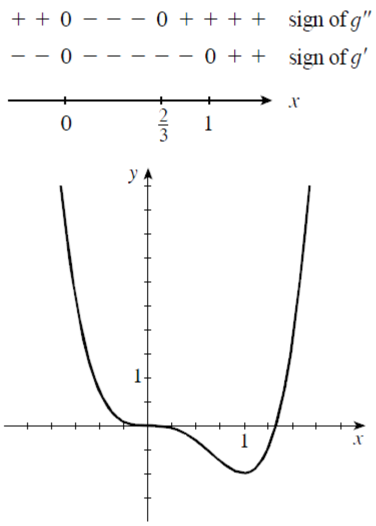
43. $\lim_{x\to-\infty}\frac{1}{2x+3}=\lim_{x\to\infty}\frac{1}{2x+3}=0$ and so $y=0$ is a horizontal asymptote. Since the denominator is equal to zero at $x=-\frac{3}{2}$ but the numerator is not, we see that $x=-\frac{3}{2}$ is a vertical asymptote.

47. $g(x)=3x^4-4x^3$

(1) The domain of g is $(-\infty,\infty)$. **(2)** $g(x)=x^3(3x-4)=0 \Leftrightarrow x=0$ or $\frac{4}{3}$, the x-intercepts. The y-intercept is 0. **(3)** There is no symmetry. **(4)** $\lim_{x\to-\infty}g(x)=\lim_{x\to\infty}g(x)=\infty$ **(5)** There is no asymptote. **(6)** $g'(x)=12x^3-12x^2=12x^2(x-1)=0\Leftrightarrow x=0$ or $x=1$, the critical numbers of g. From the sign diagram for g', we see that g is decreasing on $(-\infty,1)$ and increasing on $(1,\infty)$. **(7)** g has a relative minimum at $(1,-1)$.

(8) $g''(x)=36x^2-24x=12x(3x-2)=0\Leftrightarrow x=0$ or $\frac{2}{3}$. From the sign diagram of g'', we see that g is concave upward on $(-\infty,0)$ and $\left(\frac{2}{3},\infty\right)$ and concave downward on $\left(0,\frac{2}{3}\right)$. **(9)** g has inflection points at $(0,0)$ and $\left(\frac{2}{3},-\frac{16}{27}\right)$.

(10)



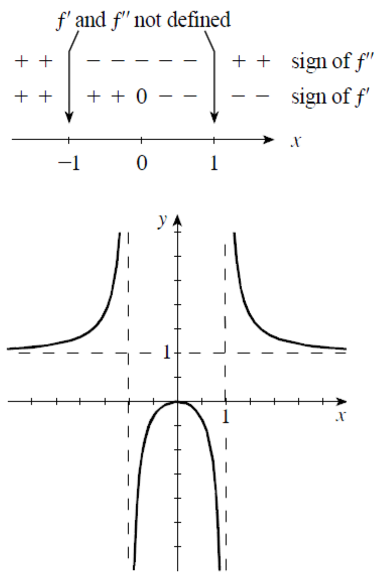
51. $f(x)=\frac{x^2}{x^2-1}$

(1) The domain of f is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$. **(2)** The x- and y-intercepts are 0. **(3)** $f(-x)=\frac{(-x)^2}{(-x)^2-1}=\frac{x^2}{x^2-1}=f(x)$, so there is symmetry with respect to the y-axis.

(4) $\lim_{x\to-\infty}f(x)=\lim_{x\to\infty}f(x)=1$ **(5)** $x=\pm1$ are vertical asymptotes. From (4), we see that $y=1$ is a horizontal asymptote.

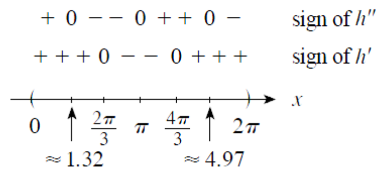
(6) $f'(x)=\frac{\left(x^2-1\right)(2x)-x^2(2x)}{\left(x^2-1\right)^2}=-\frac{2x}{\left(x^2-1\right)^2}$. From the sign diagram for f', we see that f is increasing on $(-\infty,-1)$ and $(-1,0)$ and decreasing on $(0,1)$ and $(1,\infty)$. **(7)** f has a relative maximum at $(0,0)$.

(10)



(8) $f''(x)=-2\frac{d}{dx}\frac{x}{\left(x^2-1\right)^2}=-2\left[\frac{\left(x^2-1\right)^2-x(2)\left(x^2-1\right)(2x)}{\left(x^2-1\right)^4}\right]=\frac{2\left(3x^2+1\right)}{\left(x^2-1\right)^3}$. From the sign diagram of f'', we see that f is concave upward on $(-\infty,-1)$ and $(1,\infty)$ and concave downward on $(-1,1)$ **(9)** f has no inflection point because ±1 lie outside the domain of f.

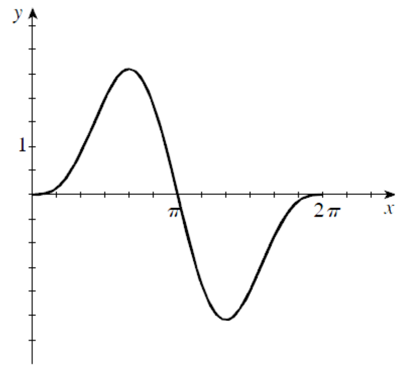
55. $h(x) = 2\sin x - \sin 2x$, $0 \le x \le 2\pi$



(1) The domain of h is $[0, 2\pi]$. **(2)** $h(0) = 0$, so the y-intercept is 0. There are x-intercepts at 0, π, and 2π. **(3)** There is no symmetry. **(4)** Not applicable **(5)** There is no asymptote.

(6) $h'(x) = 2\cos x - 2\cos 2x = 2\cos x - 2\left(2\cos^2 x - 1\right) = 0 \Leftrightarrow$ $2\cos^2 x - \cos x - 1 = (2\cos x + 1)(\cos x - 1) = 0 \Rightarrow x = \frac{2\pi}{3}, \frac{4\pi}{3}$, or π. From the sign diagram for h', we see that h is increasing on $\left(0, \frac{2\pi}{3}\right)$ and $\left(\frac{4\pi}{3}, 2\pi\right)$ and decreasing on $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$. **(7)** h has a relative maximum at $\left(\frac{2\pi}{3}, \frac{3\sqrt{3}}{2}\right)$ and a relative minimum at $\left(\frac{4\pi}{3}, -\frac{3\sqrt{3}}{2}\right)$.

(10)



(8) $h''(x) = -2\sin x + 4\sin 2x = -2\sin x + 8\sin x\cos x = 0 \Leftrightarrow$ $\sin x(1 - 4\cos x) = 0 \Leftrightarrow x = \cos^{-1}\frac{1}{4} \approx 1.32$, π, or $2\pi - \cos^{-1}\frac{1}{4} \approx 4.97$. From the sign diagram of h'', we see that h is concave upward on approximately $(0, 1.32)$ and $(\pi, 4.97)$ and concave downward on approximately $(1.32, \pi)$ and $(4.97, 2\pi)$. **(9)** h has inflection points at approximately $(1.32, 1.46)$ and $(4.97, -1.44)$ and exactly $(\pi, 0)$.